Analyse de quelques problèmes aux limites elliptiques non linéaires

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Introduction

Les travaux présentés dans ce mémoire portent sur certaines classes d'équations ou d'inéquations elliptiques non linéaires et s'organisent autour de quatre thèmes principaux: l'étude de propriétés qualitatives de solutions des problèmes elliptiques semi-linéaires et quasi-linéaires; l'analyse de solutions qui explosent à la frontière pour quelques classes de problèmes elliptiques semi-linéaires; l'étude de problèmes elliptiques non lisses via les théories de point critique de Clarke et Degiovanni, ainsi que le rôle des inégalités hemivariationnelles dans le traitement de quelques problèmes de mécanique.

Dans la première partie sont présentés quelques résultats qualitatifs pour diverses classes d'équations aux dérivées partielles elliptiques. On prouve en particulier l'existence et l'unicité de l'état fondamental pour un problème dans $\mathbf{R}^N$, l'existence d'une solution pour une classe de problèmes avec donnée au bord singulière, un résultat de multiplicité pour un problème sous critique sans compacité soumis à une petite perturbation, ainsi que des résultats d'existence ou de non existence pour un problème de bifurcation avec une donnée non linéaire sur le bord.

La deuxième partie porte sur l'étude de quelques classes de problèmes elliptiques semi-linéaires sur un domaine quelconque qui admettent des solutions explosant au bord (ou à l'infini). Notre analyse inclut l'équation logistique avec un coefficient qui peut s'annuler sur la frontière; dans ce cas on établit une condition nécessaire et suffisante pour l'existence d'une solution explosant au bord ainsi que l'unicité et le comportement asymptotique de cette solution.

Le troisième chapitre est consacré à l'étude de quelques problèmes elliptiques multivoques. On démontre un résultat abstrait de multiplicité du type Ljusternik-Schnirelmann avec application dans le problème du pendule forcé. Ensuite on étudie parallèlement un problème sans compacité en utilisant les approches de Degiovanni et de Clarke. On considère aussi un problème symétrique soumis à une contrainte et avec une infinité de solutions et on étudie l'effet d'une perturbation arbitraire en montrant que le nombre de solutions tend vers l'infini si la perturbation tend vers zéro. On étudie également plusieurs résultats d'existence et de multiplicité pour des inégalités hemivariationnelles dans $BV(\Omega; \mathbf{R}^N)$. On conclut ce chapitre avec un résultat de multiplicité et de perturbation pour une inégalité variationnelle qui modélise le démarrage d'un tremblement de terre.

Dans la dernière partie de ce mémoire on analyse quelques problèmes issus de la théorie des inégalités hemivariationnelles. On prouve des résultats d'existence du type Hartman-Stampacchia et on étudie l'influence d'une perturbation arbitraire pour plusieurs classes de problèmes aux valeurs propres pour des inégalités hemivariationnelles avec symétrie.

Tous ces problèmes conduisent à des équations, systèmes ou inéquations posés dans des domaines bornés ou non bornés. On cherche des estimations *a priori* et des théorèmes de compacité et les principaux outils appliqués sont la méthode de sur et sous solutions, le principe du maximum et la théorie du point critique.

1 Problèmes elliptiques semi-linéaires et quasi-linéaires: existence et unicité des solutions

1.1 Existence et unicité de l'état fondamental pour un problème elliptique avec une non linéarité singulière

On considère le problème

$$\begin{cases} -\Delta u = p(x)f(u) \quad \text{dans } \mathbf{R}^N \\ u > 0 \quad \text{dans } \mathbf{R}^N \\ u(x) \to 0 \quad \text{si} \quad |x| \to \infty, \end{cases} \tag{1.1}$$

où $N > 2$ et les fonctions p et f satisfont les hypothèses:
(p1) $p \in C^{\alpha}_{\text{loc}}(\mathbf{R}^N)$, $\alpha \in (0,1)$;
(p2) $p > 0$ dans $\mathbf{R}^N$.
(p3) $\int_0^\infty r \cdot \Phi(r)dr < \infty$, où $\Phi(r) = \max_{|x|=r} p(x)$.
(f1) il existe $\beta > 0$ tel que l'application $u \longmapsto \dfrac{f(u)}{u+\beta}$ soit décroissante sur $(0,\infty)$;
(f2) $\lim_{u \searrow 0} \dfrac{f(u)}{u} = +\infty$ et f est bornée dans un voisinage de $+\infty$.

Théorème 1 *Sous les hypothèses (f1), (f2), (p1)-(p3), le problème (1.1) a une seule solution $u \in C^{2+\alpha}_{\text{loc}}(\mathbf{R}^N)$.*

Le résultat suivant montre le rôle de la condition (p3) en vue de l'existence d'une solution.

Théorème 2 *Soit p une fonction positive, radiale, continue sur $\mathbf{R}^N$ et telle que*

$$\int_0^\infty rp(r)dr = \infty.$$

Alors le problème (1.1) n'a aucune solution positive radiale.

1.2 Solutions multiples d'un problème sous-critique dégénéré

Soient $N \geq 2$ et $2 < p < 2^* := 2N/(N-2)$. On considère le problème

$$-\text{div}\,(a(x)\nabla u) + b(x)\,u = K(x)|u|^{p-2}u + \varepsilon\, g(x) \quad \text{dans } \mathbf{R}^N, \tag{1.2}$$

où les fonctions a, b, K et g satisfont les hypothèses

$(A1)$ $\quad a \in C(\mathbf{R}^N)$ et il existe $R_0 > 0$ tel que
$\{x;\, a(x) = 0\} \subset B(0, R_0)$ et $\dfrac{1}{a} \in L^q(B(0, R_0))$, pour un certain
$q > \dfrac{Np}{2N + 2p - Np}$;

$(A2)$ $\quad \lim\limits_{|x|\to\infty} a(x) = a(\infty) \in \mathbf{R}_+$ et $0 \le a(x) \le a(\infty)$ dans $\mathbf{R}^N$;

(B) $\quad$ ess $\lim\limits_{|x|\to\infty} b(x) = b(\infty) \in \mathbf{R}_+$ et il existe $b_1 > 0$ tel que $b_1 \le b(x) \le b(\infty)$ a.e. $\mathbf{R}^N$

(K) $\quad$ ess $\lim\limits_{|x|\to\infty} K(x) = K(\infty) \in \mathbf{R}_+$ et $K(x) \ge K(\infty)$ a.e. dans $\mathbf{R}^N$;

(M) $\quad$ meas $(\{x \in \mathbf{R}^N;\, b(x) < b(\infty)\} \cup \{x \in \mathbf{R}^N;\, K(x) > K(\infty)\}) > 0$.

Soit E le complété de $C_0^\infty(\mathbf{R}^N)$ par rapport à la norme

$$\|u\|_E^2 = \int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\, dx.$$

On suppose que $g \in E^*$, $g \neq 0$.

L'effet d'une petite perturbation pour un problème elliptique semi-linéaire a été étudié dans Tarantello [49], où il est montré que si ε est assez petit, alors le problème (1.2) admet au moins deux solutions dans le cas critique $p = 2^*$ et non-dégénéré $a = 1$, pour un domaine borné, avec $b = 0$ et $K = 1$.

En utilisant le principe variationnel d'Ekeland ainsi que le théorème du col sans la condition de Palais-Smale (voir Brezis-Nirenberg [6, Theorem 2.2]) combiné avec une variante du lemme de Brezis-Lieb [5] on montre

THÉORÈME 3 *Sous les hypothèses (A1), (A2), (B), (K) et (M), il existe* $\varepsilon_0 > 0$ *tel que le problème (1.2) admet au moins deux solutions, si* $0 < \varepsilon < \varepsilon_0$.

1.3 Solutions multiples d'un problème critique dégénéré

Soient Ω un ouvert de $\mathbf{R}^N$, $N \ge 2$ et $\alpha \in (0, 2)$. On désigne par $H_0^1(\Omega; |x|^\alpha)$ la fermeture de $C_c^\infty(\Omega)$ par rapport à la norme

$$||\zeta||_\alpha = \left(\int_\Omega |x|^\alpha |\nabla\zeta|^2 dx\right)^{1/2} \qquad \forall \zeta \in C_c^\infty(\Omega).$$

Soit $H^{-1}(\Omega; |x|^\alpha)$ l'espace dual de $H_0^1(\Omega; |x|^\alpha)$ et soit E_+ le cône positif de $E = H^{-1}(\Omega; |x|^\alpha)$. On considère le problème

$$(1.3) \qquad \begin{cases} -\mathrm{div}\,(|x|^\alpha \nabla u) = |u|^{2_\alpha^* - 2} u + f & \text{dans} \quad \Omega, \\ u \ge 0, \;\; u \not\equiv 0 & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \end{cases}$$

où $2_\alpha^* = 2N/(N - 2 + \alpha)$ et $f \in H^{-1}(\Omega; |x|^\alpha)$. On observe que ce problème devient dégénéré si $0 \in \overline{\Omega}$ ou si Ω est non-borné.

On définit

$$s_\alpha^0(\Omega) = \lim_{r\to 0} S_\alpha(\Omega \cap B_r) \qquad s_\alpha^\infty(\Omega) = \lim_{r\to\infty} S_\alpha(\Omega \setminus B_r).$$

On dit que $\Omega \subset \mathbf{R}^N$ satisfait la condition $\mathcal{C}$ si Ω est un cône de $\mathbf{R}^N$, ou $\Omega = \mathbf{R}^N$, ou

$$S_\alpha(\Omega) < \min\{s_\alpha^0(\Omega), s_\alpha^\infty(\Omega)\}.$$

On démontre le résultat suivant de multiplicité.

THÉORÈME 4 *Supposons que Ω satisfait la condition $\mathcal{C}$. Alors, pour chaque $g \in E_+$, il existe $\varepsilon_0 > 0$ tel que pour chaque $0 < \varepsilon \leq \varepsilon_0$, le problème (1.3) avec $f = \varepsilon g$ admet au moins deux solutions.*

1.4 Problèmes quasi-linéaire avec condition aux limites non linéaire

Soit $\Omega \subset \mathbf{R}^N$ un domaine non-borné avec frontière régulière Γ et soit n le vecteur unité de la normale extérieure sur Γ. On considère le problème aux limites

$$(A) \qquad \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x)|u|^{p-2}u + g(x)|u|^{q-2}u & \text{dans } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)\cdot|u|^{p-2}u = h(x,u) & \text{sur } \Gamma, \end{cases}$$

où $p < q < p^* = \frac{Np}{N-p}$ si $p < N$ ($p^* = +\infty$ si $p \geq N$), $0 < a_0 \leq a \in L^\infty(\Omega)$ et $b : \Gamma \to \mathbf{R}$ est une fonction continue telle que

$$\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}},$$

où c et C sont des constantes positives.

Soit $h : \Gamma \times \mathbf{R} \to \mathbf{R}$ une fonction de Carathéodory telle que
(A1) $|h(x,s)| \leq h_0(x) + h_1(x)|s|^{m-1}$; $q \leq m < \frac{(N-1)p}{N-p}$ si $p < N$ ($q \leq m < +\infty$ si $p \geq N$),
où $h_i : \Gamma \to \mathbf{R}$ ($i = 0,1$) sont des fonctions mesurable qui satisfont $h_0 \in L^{m/(m-1)}(\Gamma;\, w_3^{1/(1-m)})$,

$$0 \leq h_i \leq C_h w_3 \qquad \text{p.p. dans } \Gamma,$$

où $C_h > 0$, avec $w(x) = (1+|x|)^\alpha$, $x \in \Gamma$ et $-N < \alpha < m\cdot\frac{N-p}{p} - N + 1$ si $p < N$ ($-N < \alpha < 0$ si $p \geq N$).

On suppose aussi que
(A2) $\displaystyle\lim_{s\to 0}\frac{h(x,s)}{b(x)|s|^{p-1}} = 0$ uniformément en x
(A3) il existe $\mu \in (p,q]$ tel que

$$\mu H(x,s) \leq s h(x,s) \text{ p.p. } x \in \Gamma \text{ et pour chaque } s \in \mathbf{R}$$

(A4) il existe un ouvert $\emptyset \neq U \subset \Gamma$ tel que $H(x,s) > 0$ pour $(x,s) \in U \times (0,\infty)$, où $H(x,s) = \int\limits_0^s h(x,t)\,dt$.

Soit $C_\delta^\infty(\Omega)$ l'espace des fonctions $C_0^\infty(\mathbf{R}^N)$ restreintes à Ω et soit E le complété de $C_\delta^\infty(\Omega)$ par rapport à la norme

$$\|u\|_E = \left(\int\limits_\Omega \left(|\nabla u(x)|^p + \frac{1}{(1+|x|)^p}|u(x)|^p\right)dx\right)^{1/p}.$$

Soit

$$\tilde{\lambda} := \inf_{u \in E;\ u \neq 0} \left(\frac{\int\limits_{\Omega} a(x)|\nabla u|^p\,dx + \int\limits_{\Gamma} b(x)|u|^p\,d\Gamma}{\int\limits_{\Omega} f(x)|u|^p\,dx} \right).$$

On montre les résultats suivants

THÉORÈME 5 *Supposons que les conditions (A1)-(A4) soient satisfaites. Alors, pour chaque $\lambda < \tilde{\lambda}$, le problème (A) admet au moins une solution non-triviale.*

THÉORÈME 6 *Supposons $h(x,s) = 0$ et $q \geq 2$. Alors, pour chaque $\lambda < \tilde{\lambda}$, le problème (A) admet une infinité de solutions.*

On considère maintenant le problème non linéaire aux valeurs propres

$$(B) \qquad \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x)|u|^{p-2}u + g(x)|u|^{q-2}u],\ \text{dans } \Omega \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = \lambda h(x,u),\ \text{sur } \Gamma. \end{cases}$$

On montre

THÉORÈME 7 *Supposons que les hypothèses (A1) et (A3) soient satisfaites. Soit d un réel tel que $1/d$ n'est pas une valeur propre λ pour le problème (B) et satisfaisant*

$$d > \frac{1}{\tilde{\lambda}}.$$

Alors il existe $\overline{\rho} > 0$ tel que pour chaque $r > \rho \geq \overline{\rho}$, le problème (B) admet une solution $(u,\lambda) = (u_d, \lambda_d) \in E \times \mathbf{R}$, telle que

$$\lambda_d \in \left[\frac{1}{d + r^2\,\|u_d\|_b^{m-p}}, \frac{1}{d + \rho^2\,\|u_d\|_b^{m-p}} \right].$$

1.5 Résultats d'existence et de non-existence pour un problème quasi-linéaire avec conditions aux limites non linéaires

Soit $\Omega \subset \mathbf{R}^N$ un domaine non-borné avec frontière régulière Γ et soit n le vecteur unité de la normale extérieure sur Γ. On considère le problème

$$(1_{\lambda,\theta}) \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) + h(x)u^{r-1} = f(\lambda, x, u) \quad \text{dans } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)\cdot u^{p-1} = \theta g(x,u) \quad \text{sur } \Gamma, \\ u \geq 0,\ u \not\equiv 0 \quad \text{dans } \Omega, \end{cases}$$

où

$$1 < p < N, \quad \max\{p, 2\} < r < p^\star := \frac{pN}{N-p}$$

$$a \in L^\infty(\Omega),\ a(x) \geq a_0 > 0 \qquad \text{p.p. } x \in \Omega$$

$$\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}}, \qquad \text{p.p. } x \in \Gamma, \text{ où } c,\ C > 0.$$

$$h : \Omega \to (0,\infty) \text{ est continue et } \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx < \infty.$$

On suppose que $g : \Gamma \times \mathbf{R} \to \mathbf{R}$ est une fonction de Carathéodory telle que
(g1) $g(\cdot,0) = 0$, $g(x,s) + g(x,-s) \geq 0$ p.p. $x \in \Gamma$ et pour chaque $s \in \mathbf{R}$;
(g2) $|g(x,s)| \leq g_0(x) + g_1(x)|s|^{m-1}$; $p \leq m < p \cdot \frac{N-1}{N-p}$, où g_i sont non-négatives, mesurables et telles que

$$0 \leq g_i(x) \leq C_g(1+|x|)^{\alpha_2} \text{ p.p.}, \quad g_0 \in L^{m/(m-1)}(\Gamma;\, w_2^{1/(1-m)}),$$

où $-N < \alpha_2 < m \cdot \frac{N-p}{p} - N + 1$.

On suppose que $f(\lambda,x,s) : (0,\infty) \times \Omega \times \mathbf{R} \to \mathbf{R}$ est une fonction croissante en λ, mesurable en x, dérivable en s et qui satisfait
(f1) $f(\cdot,\cdot,0) = 0$, $f(\lambda,x,s) + f(\lambda,x,-s) \geq 0$ $\forall \lambda > 0$, p.p. $x \in \Omega$, $\forall s \in \mathbf{R}$;
(f2) $|f_s(\lambda,x,s)| \leq \lambda \phi(x)|s|^{q-2}$ avec $r > q > \max\{p,2\}$, $\forall \lambda > 0$, p.p. $x \in \Omega$, $\forall s \in \mathbf{R}$, où $\phi \geq 0$ est une fonction mesurable telle que

$$0 \leq \phi(x) \leq c_f\, w_1(x) \quad \text{p.p. } x \in \Omega\,;$$

(f3) $\lim_{s\to 0} \dfrac{f(\lambda,x,s)}{\lambda w_1(x)|s|^{q-2}s} = 1$ uniformément par rapport à x et λ;
(f4) $|f(\lambda_1,x,s) - f(\lambda_2,x,s)| \leq |\lambda_1 - \lambda_2|\psi(x)|s|^{q-1}$, $\forall \lambda_1,\lambda_2 > 0$, p.p. $x \in \Omega$, $\forall s \in \mathbf{R}$, où $\psi \geq 0$ est une fonction mesurable telle que

$$0 \leq \psi(x) \leq C_f\, w_1(x) \quad \text{p.p. } x \in \Omega\,.$$

On démontre que, sous ces hypothèses, ont lieu les résultats suivants

THÉORÈME 8 *Il existe θ_*, θ^* et $\lambda^* > 0$ tels que le problème $(1_{\lambda,\theta})$ n'admet aucune solution si $\theta_* < \theta < \theta^*$ et $0 < \lambda < \lambda^*$.*

On définit maintenant

$$U = \{u \in X : \int_\Gamma G(x,u)\,d\Gamma < 0\}, \qquad V = \{u \in X : \int_\Gamma G(x,u)\,d\Gamma > 0\}$$

et

$$\theta_- = \sup_{u\in U} \frac{\|u\|_b^p}{p\int_\Gamma G(x,u)\,d\Gamma}, \qquad \theta^+ = \inf_{u\in V} \frac{\|u\|_b^p}{p\int_\Gamma G(x,u)\,d\Gamma}$$

Si $U = \emptyset$ (resp. $V = \emptyset$) alors $\theta_- = -\infty$ (resp. $\theta^+ = +\infty$).

On montre

THÉORÈME 9 *Soit $\underline{\theta} = \max\{\theta_*, \theta_-\}$, $\overline{\theta} = \min\{\theta^*, \theta^+\}$ et supposons que $J = (\underline{\theta}, \overline{\theta}) \neq \emptyset$. Alors il existe $\lambda_0 > 0$ tel que*

(i) le problème $(1_{\lambda,\theta})$ admet une solution si $\lambda \geq \lambda_0$ et $\theta \in J$;

(ii) le problème $(1_{\lambda,\theta})$ n'a aucune solution si $0 < \lambda < \lambda_0$ et $\theta \in J$.

2 Problèmes elliptiques singuliers: existence, unicité et explosion des solutions

2.1 Problèmes avec donnée au bord singulière

Soit Ω un ouvert regulier de $\mathbf{R}^N$, $\Omega \neq \mathbf{R}^N$. On considère le problème

$$
(2.4) \qquad \begin{cases} \Delta u = p(x) f(u) & \text{dans } \Omega\,, \\ u \geq 0,\ u \not\equiv 0 & \text{dans } \Omega\,, \\ u(x) \to \infty & \text{si dist}\,(x, \partial\Omega) \to 0. \end{cases}
$$

Ce type de problème fait actuellement l'objet de nombreux travaux portant sur l'existence, l'unicité et le comportement asymptotique des solutions au voisinage de la frontière. Les premiers résultats d'existence ont été obtenus par Keller [31] et Osserman [40]. Ils ont prouvé que si Ω est borné, $p \equiv 1$ et $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, f croissante, $f(0) = 0$, alors une condition nécessaire et suffisante pour l'existence de solutions est

$$
\int_1^\infty F^{-1/2}(t)\, dt < +\infty, \qquad \text{où } F'(t) = f(t).
$$

Dans le cas particulier $p \equiv 1$ et $f(u) = u^{(N+2)/(N-2)}$, $N > 2$, qui apparaît dans de nombreux problèmes géométriques, Loewner et Nirenberg [32] ont étudié les questions d'unicité et de comportement asymptotique. Bandle et Marcus [2] ont étendu ces résultats à d'autres nonlinéarités, comme $f(u) = u^p$, $p > 1$.

On suppose que f satisfait

$(f1)$ $\quad f \in C^1[0, \infty),\ f' \geq 0,\ f(0) = 0$ et $f > 0$ sur $(0, \infty)$

ainsi que la condition de Keller-Osserman

$$
(f2) \qquad \int_1^\infty [F(t)]^{-1/2}\, dt < \infty\,, \quad \text{où} \quad F(t) = \int_0^t f(s)\, ds.
$$

La fonction p est continue, non-négative et peut s'annuler dans des sous ensembles de Ω qui ne touchent pas la frontière de Ω. Si Ω est borné, on suppose que

$(p1)$ pour chaque $x_0 \in \Omega$ avec $p(x_0) = 0$, il existe $x_0 \ni \Omega_0 \subset\subset \Omega$ tel que $p > 0$ sur $\partial\Omega_0$.

Dans ce cas on montre

THÉORÈME 10 *Supposons les hypothèses (f1), (f2) et (p1) soient satisfaites. Alors le problème (2.4) a au moins une solution.*

Si $\Omega = \mathbf{R}^N$ on démontre un résultat similaire, mais avec la condition (p1) remplacée par une condition adaptée au cas non-borné. Si $\Omega \neq \mathbf{R}^N$ est un ouvert non-borné, Marcus [33] a construit une solution de (2.4) qui tend vers zero à l'infini et qui, de plus, est la solution explosive **minimale** du problème (2.4). Nous montrons que si $\Omega = \mathbf{R}^N \setminus \overline{B(0, R)}$, alors le problème (2.4) admet deux types de solutions explosives, selon leur comportement à l'infini: la solution de Marcus et une autre solution U telle que $U(x) \to +\infty$ si $|x| \to \infty$. De plus, la solution U est **maximale** parmi toutes les solutions larges du problème (2.4).

2.2 Problèmes d'explosion pour l'équation logistique

Soit $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ un ouvert borné régulier. On considère le problème

$$\Delta u + au = b(x)f(u) \qquad \text{dans} \;\; \Omega, \tag{2.5}$$

où a est un réel et $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, tel que $b \geq 0$ et $b \not\equiv 0$ sur Ω. Soit

$$\Omega_0 = \operatorname{int}\{x \in \Omega : \; b(x) = 0\}$$

et on suppose que $\overline{\Omega}_0 \subset \Omega$ et que $\partial\Omega_0$ satisfait la condition du cône extérieur. La non-linéarité $f \in C^1[0, \infty)$ satisfait
(A_1) $f \geq 0$ et $f(u)/u$ est croissante sur $(0, \infty)$.
(A_2) $\displaystyle\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty$, où $F(t) = \displaystyle\int_0^t f(s)\,ds$.

Soit $\lambda_{\infty,1}$ la première valeur propre de $(-\Delta)$ dans Ω_0. On considère $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$.

Le résultat suivant donne une condition nécessaire et suffisante pour l'existence d'une solution explosive pour l'équation (2.5).

THÉORÈME 11 *Supposons que f satisfait les conditions (A_1) et (A_2). Alors le problème (2.5) admet une solution explosive positive si et seulement si $a \in (-\infty, \lambda_{\infty,1})$.*

On remarque que dans le résultat ci-dessus b peut s'annuler sur $\partial\Omega$ ou même $b \equiv 0$ sur $\partial\Omega$. On repond ainsi à une question posée par le Professeur Haim Brezis en mai 2001.

On considère maintenant le problème

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{dans} \;\; \Omega \setminus \overline{\Omega}_0\,, \\ \mathcal{B}u = 0 & \text{sur} \;\; \partial\Omega\,, \\ u = +\infty & \text{sur} \;\; \partial\Omega_0\,, \end{cases} \tag{2.6}$$

où $b > 0$ sur $\partial\Omega$ et $\mathcal{B}$ signifie une condition de Dirichlet, de Neumann ou de Robin.

On démontre que le problème (2.6) admet des solutions maximales, pour n'importe quel valeur de a. Plus précisèment, on a

THÉORÈME 12 *Pour chaque $a \in \mathbf{R}$, le problème (2.6) admet une solution maximale et une solution minimale.*

On montre aussi que, sous des hypothèses supplémentaires, le problème (2.6) admet une solution unique.

Parmi les non-linéarités qui satisfont les hypothèses de nos résultats on cite
(i) $f(u) = u^p$, $p > 1$; (ii) $f(u) = u^p \ln(u+1)$, $p > 1$; (iii) $f(u) = u^p \arctan u$, $p > 1$.

2.3 Solutions explosives à l'infini pour les systèmes elliptiques

On considère le système

$$\begin{cases} \Delta u = p(x)g(v) & \text{dans} \;\; \mathbf{R}^N, \\ \Delta v = q(x)f(u) & \text{dans} \;\; \mathbf{R}^N, \end{cases} \tag{2.7}$$

où $p, q \in C^{0,\alpha}_{\mathrm{loc}}(\mathbf{R}^N)$ $(0 < \alpha < 1)$ sont des fonctions non-négatives et à symétrie radiale. On suppose que $f, g \in C^{0,\beta}_{\mathrm{loc}}[0,\infty)$ $(0 < \beta < 1)$ sont des fonctions non-négatives et croissantes.

Soit

$$\mathcal{G} = \{(a,b) \in \mathbf{R}^+ \times \mathbf{R}^+;\ (\exists) \text{ solution radiale de (2.7) telle que } (u(0), v(0)) = (a,b)\}.$$

On montre

THÉORÈME 13 *Supposons que*

$$\lim_{t\to\infty} \frac{g(cf(t))}{t} = 0 \qquad \forall\, c > 0.$$

Alors $\mathcal{G} = \mathbf{R}^+ \times \mathbf{R}^+$.

De plus,
i) Si

$$\int_0^\infty tp(t)\,dt = \int_0^\infty tq(t)\,dt = +\infty,$$

alors toute solution radiale positive de (2.7) explose à l'infini.
ii) Si

$$\int_0^\infty tp(t)\,dt < \infty, \qquad \int_0^\infty tq(t)\,dt < \infty$$

alors toute solution radiale positive de (2.7) est bornée. Si $(\tilde{u}, \tilde{v})$ *sont deux solutions radiales positives de (2.7), alors il existe une constante* C *telle que, pour chaque* $r \in [0,\infty)$,

$$\max\{|u(r) - \tilde{u}(r)|, |v(r) - \tilde{v}(r)|\} \le C \max\{|u(0) - \tilde{u}(0)|, |v(0) - \tilde{v}(0)|\}.$$

Supposons maintenant que $f, g \in C^1[0,\infty)$ satisfont

$(\mathbf{H_1})$ $\quad f(0) = g(0) = 0,\ \liminf_{u\to\infty} \frac{f(u)}{g(u)} =: \sigma > 0$

ainsi que la condition de Keller-Osserman

$(\mathbf{H_2})$ $\quad \displaystyle\int_1^\infty \frac{dt}{\sqrt{G(t)}} < \infty, \text{ où } G(t) = \int_0^t g(s)\,ds.$

Dans ce cas on montre

THÉORÈME 14 *Soient* $f, g \in C^1[0,\infty)$ *qui satisfont* $(\mathbf{H_1})$ *et* $(\mathbf{H_2})$. *Si*

$$\int_0^\infty tp(t)\,dt < \infty, \qquad \int_0^\infty tq(t)\,dt < \infty,$$

alors toute solution radiale (u, v) *de (2.7) avec* $(u(0), v(0)) \in F(\mathcal{G})$ *est une solution explosive à l'infini.*

2.4 Unicité de la solution explosant au bord pour équations logistiques avec absorption

Soit $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) un domaine borné et régulier, a un paramètre réel et $b \in C^{0,\mu}(\overline{\Omega})$, $\mu \in (0,1)$, $b \geq 0$, $b \not\equiv 0$ dans Ω. On considère l'équation logistique

$$\Delta u + au = b(x) f(u) \qquad \text{dans } \Omega, \tag{2.8}$$

où $f \in C^1[0,\infty)$ satisfait

(A_1) $\quad f \geq 0$ et $f(u)/u$ est strictement croissante sur $(0,+\infty)$.

Soit

$$\Omega_0 := \text{int}\,\{x \in \Omega : \; b(x) = 0\}$$

et on suppose que $\partial\Omega_0$ est régulier (éventuellement vide), $\overline{\Omega}_0 \subset \Omega$ et $b > 0$ sur $\Omega \setminus \overline{\Omega}_0$. On désigne par $\lambda_{\infty,1}$ la première valeur propre (avec conditions de Dirichlet) de l'opérateur $(-\Delta)$ dans Ω_0, avec la convention $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$.

On dit que u est une solution *large (explosive)* de (2.8) si $u \geq 0$ dans Ω et $u(x) \to \infty$ si $d(x) := \text{dist}\,(x, \partial\Omega) \to 0$.

Soit $D > 0$ et $R : [D,\infty) \to (0,+\infty)$ une fonction mesurable. On dit que R a une variation régulière d'indice $\rho \in \mathbf{R}$ (notation: $R \in \mathbf{R}_\rho$) si $\lim_{u\to\infty} R(\xi u)/R(u) = \xi^\rho$, pour chaque $\xi > 0$.

Soit $\mathcal{K}$ l'ensemble des fonctions $k : (0,\nu) \to (0,+\infty)$ (pour un certain ν), de classe C^1, croissantes, telles que $\lim_{t\to 0^+} \left(\dfrac{\int_0^t k(s)\,ds}{k(t)} \right)^{(i)} := \ell_i$, pour $i = \overline{0,1}$.

On démontre le résultat suivant.

Théorème 15 *Supposons que la fonction f satisfait la condition (A_1) et que f' est une fonction à variation régulière d'indice $\rho \neq 0$. De plus, on suppose que le potentiel b vérifie*

(B) $\quad b(x) = c\,k^2(d(x)) + o(k^2(d(x)))$ *si* $d(x) \to 0$, *avec* $c > 0$ *et* $k \in \mathcal{K}$.

Alors, pour chaque $a \in (-\infty, \lambda_{\infty,1})$, l'équation (2.8) admet une unique solution explosive u_a. On a, de plus,

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0,$$

où $\xi_0 = \left(\dfrac{2 + \ell_1 \rho}{c(2+\rho)} \right)^{1/\rho}$ *et la fonction h est définie par*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0,\nu).$$

2.5 Comportement asymptotique de la solution explosant au bord pour l'équation logistique avec absorption

On continue l'étude du problème logistique

$$\Delta u + au = b(x) f(u) \;\text{ dans } \Omega, \qquad u(x) \to +\infty \;\text{ si } d(x) := \text{dist}\,(x, \partial\Omega) \to 0, \tag{2.9}$$

sous les hypothèses de la section précédente.

Soit $\mathcal{K}$ l'ensemble des fonctions $k : (0,\nu) \to (0,\infty)$ (pour un certain ν), de classe C^1, croissantes, telles que $\lim_{t\searrow 0}(\int_0^t k(s)\,ds/k(t))^{(i)} := \ell_i$, pour $i = \overline{0,1}$.

Soit RV_q ($q \in \mathbf{R}$) l'ensemble des fonctions positives et mesurables $Z : [A,\infty) \to \mathbf{R}$ (avec $A > 0$) telles que $\lim_{u\to\infty} Z(\xi u)/Z(u) = \xi^q$, $\forall \xi > 0$. On désigne par NRV_q la classe des fonctions f définies par $f(u) = Cu^q \exp\{\int_B^u \phi(t)/t\,dt\}$, $\forall u \geq B > 0$, où $C > 0$ et $\phi \in C[B,\infty)$ satisfait $\lim_{t\to\infty} \phi(t) = 0$. Supposons que $0 \leq f \in C^1[0,\infty) \cap NRV_{\rho+1}$ ($\rho > 0$) est telle que $f(u)/u$ soit strictement croissante sur $(0,\infty)$ et que $b \equiv 0$ sur $\partial\Omega$ vérifie $b(x) = k^2(d)(1+o(1))$ si $d(x) \to 0$, avec $k \in \mathcal{K}$. Alors, pour chaque $a < \lambda_{\infty,1}$, le problème (1) admet une unique solution positive u_a (voir Théorème 15).

Pour chaque $\zeta > 0$, soit

$$\mathcal{R}_{0,\zeta} = \left\{ \begin{array}{ll} k: & k(u^{-1}) = d_0 u\,[\Lambda(u)]^{-1} \exp\left[-\int_{d_1}^u (s\Lambda(s))^{-1}\,ds\right] \ (u \geq d_1),\ 0 < \Lambda \in C^1[d_1,\infty), \\ & \lim_{u\to\infty} \Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0,\ \lim_{u\to\infty} u^{\zeta+1}\Lambda'(u) = \ell_\star \in \mathbf{R},\ d_0,\ d_1 > 0 \end{array} \right\}.$$

On a $\mathcal{R}_{0,\zeta} \subset \mathcal{K}$. De plus, si $k \in \mathcal{R}_{0,\zeta}$ alors $\ell_1 = 0$ et $\lim_{t\to 0} k(t) = 0$.

On définit les classes $\mathcal{F}_{\rho\eta} = \{f \in NRV_{\rho+1}(\rho > 0) :\ \phi \in RV_\eta \text{ ou } -\phi \in RV_\eta\}$, si $\eta \in (-\rho-2, 0]$ et $\mathcal{F}_{\rho 0,\tau} = \{f \in \mathcal{F}_{\rho 0} :\ \lim_{u\to\infty} (\ln u)^\tau \phi(u) = \ell^\star \in \mathbf{R}\}$, pour $\tau \in (0,\infty)$.

On démontre le résultat suivant.

Théorème 16 *On suppose que $b(x) = k^2(d)(1 + \tilde{c}d^\theta + o(d^\theta))$ si $d(x) \to 0$ (avec $\theta > 0$, $\tilde{c} \in \mathbf{R}$), où $k \in \mathcal{R}_{0,\zeta}$. Soit $0 \leq f \in C^1[0,\infty)$ telle que $f(u)/u$ soit strictement croissante sur $(0,\infty)$. De plus, on suppose que f satisfait l'un des cas suivants de croissance à l'infini:*

(i) $f(u) = Cu^{\rho+1}$ *dans un voisinage de l'infini;*

(ii) $f \in \mathcal{F}_{\rho\eta}$ *avec* $\eta \neq 0$*;*

(iii) $f \in \mathcal{F}_{\rho 0,\tau_1}$ *avec* $\tau_1 = \varpi/\zeta$, *où* $\varpi = \min\{\theta,\zeta\}$.

Alors, pour chaque $a \in (-\infty, \lambda_{\infty,1})$, l'unique solution positive u_a du problème (2.9) satisfait

$$u_a(x) = \xi_0 h(d)(1 + \chi d^\varpi + o(d^\varpi)) \quad \text{si} \ \ d(x) \to 0,$$

où $\xi_0 = [2(2+\rho)^{-1}]^{1/\rho}$ et h est définie par $\int_{h(t)}^\infty [2F(s)]^{-1/2}ds = \int_0^t k(s)ds$, pour $t > 0$ suffisement petit. L'expression de χ est donnée par

$$\chi = \left\{ \begin{array}{l} -(1+\zeta)\ell_\star(2\zeta)^{-1}\text{Heaviside}(\theta-\zeta) - \tilde{c}\rho^{-1}\text{Heaviside}(\zeta-\theta) = \chi_1,\ \textit{pour (i) et (ii)} \\ \chi_1 - \ell^\star\rho^{-1}\left(-\rho\ell_\star/2\right)^{\tau_1}\left(1/(\rho+2) + \ln\xi_0\right),\ \textit{pour le cas (iii)}. \end{array} \right.$$

2.6 Problèmes elliptiques singuliers anisotropes

On considère le problème

$$\text{(2.10)} \qquad \left\{ \begin{array}{ll} \displaystyle\sum_{i=1}^{N-1} f_i(u)\,u_{x_i x_i} + u_{yy} + p(x)\,g(u) = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{array} \right.$$

où $\Omega \subset \mathbf{R}^N$ est un ouvert borné régulier et p est une fonction continue positive sur $\overline{\Omega}$.

On suppose que

$(\mathbf{H}_1)$ $f_i, g : (0,\infty) \to (0,\infty)$, $i = \overline{1, N-1}$ sont de classe C^1;

$(\mathbf{H}_2)$ f_i, $i = \overline{1, N-1}$ est croissante sur $(0,\infty)$ et g est décroissante sur $(0,\infty)$.

Soit

$$D = \{y \in [0,\ell] : \ \exists x' = (x_1, \ldots, x_{N-1}) \text{ tel que } \ (x', y) \in \overline{\Omega}\}.$$

Soit ψ l'unique fonction positive définie par

$$\int_0^{\psi(y)} \frac{1}{g(t)}\, dt = \frac{\beta}{2}(\ell y - y^2), \quad \text{pour chaque } \ y \in [0,\ell].$$

Alors

$$\max_{y \in D} \psi(y) \leq \max_{y \in [0,\ell]} \psi(y) =: A,$$

où $A > 0$ est défini implicitement par

$$\int_0^A \frac{1}{g(t)}\, dt = \frac{\beta}{8}\ell^2.$$

On impose aussi

$(\mathbf{H}_3)$ $f_1' > 0$ sur $(0, A]$.

$(\mathbf{C}_1)$ il existe $\lim\limits_{x \searrow 0} \dfrac{f_1 f_i'}{f_1'}(x) \in \mathbf{R}$, pour chaque $i = \overline{2, N-1}$.

Pour $x \in \Omega$ on définit les ensembles

$$P_x = \{2 \leq i \leq N-1;\ u_{x_i x_i}(x) \geq 0\} \quad \text{et} \quad N_x = \{2 \leq i \leq N-1;\ u_{x_i x_i}(x) < 0\}.$$

Notre premier résultat donne une condition suffisante pour l'unicité de la solution, si on savait qu'au moins une solution existerait.

THÉORÈME 17 *Supposons que les hypothèses* $(\mathbf{H}_1)$-$(\mathbf{H}_3)$ *et* $(\mathbf{C}_1)$ *soient satisfaites. Alors il existe* $K_1 = K_1(f_1, g, p, \Omega)$ *tel que si* u *est une solution positive de (2.10) qui satisfait*

$$\sum_{i \in P_x} m_i u_{x_i x_i} + \sum_{i \in N_x} M_i u_{x_i x_i} + u_{yy} > -K_1 \quad \text{sur } \Omega$$

alors u *est l'unique solution de (2.10).*

Dans la suite on impose la condition

$(\mathbf{C}_2)$ $\dfrac{f_i}{f_1}$, $i = \overline{2, N-1}$ est décroissante sur $(0,\infty)$.

Dans ce cas on montre

THÉORÈME 18 *Supposons que les hypothèses* $(\mathbf{H}_1)$-$(\mathbf{H}_3)$ *et* $(\mathbf{C}_2)$ *soient satisfaites. Alors il existe* $K_2 = K_2(f_1, g, p, \Omega)$ *tel que si* u *est une solution positive de (2.10) qui, de plus, satisfait*

$$u_{x_1 x_1} + \sum_{i \in P_x} \frac{f_i(u)}{f_1(u)} u_{x_i x_i} + \sum_{i \in N_x} \left(\inf_{(0,A)} \frac{f_i'}{f_1'} \right) u_{x_i x_i} < K_2 \quad \text{sur } \Omega$$

alors u *est l'unique solution de (2.10).*

3 Problèmes elliptiques non lisses: théories de Clarke et de Degiovanni

3.1 Un résultat de multiplicité pour des fonctionnelles localement Lipschitz périodiques

Soit X un espace de Banach réel et $f : X \to \mathbf{R}$ une application localement Lipschitz. Soit

$$\partial f(u) = \{x^* \in X^*;\ \ f^0(u;v) \geq \ \langle x^*, v\rangle,\ \text{pour tout } v \in X\}$$

le gradient de Clarke au point $u \in X$, où

$$f^0(u;v) = \limsup_{\substack{w \to u \\ \lambda \searrow 0}} \frac{f(w + \lambda v) - f(w)}{\lambda}, \qquad v \in X\,.$$

Le point $u \in X$ est un point critique de f si $0 \in \partial f(u)$, c'est-à-dire $f^0(u;v) \geq 0$, pour chaque $v \in X$.

Soit Z un sous-groupe discret de X, donc $\inf_{z \in Z \setminus \{0\}} \|z\| > 0$.

Une fonction $f : X \to \mathbf{R}$ est *Z-périodique* si $f(x+z) = f(x)$, pour chaque $x \in X$ et $z \in Z$.

Si $\pi : X \to X/Z$ est la surjection canonique est si x est un point critique de f, alors l'ensemble $\pi^{-1}(\pi(x))$ ne contient que de points critiques de f. L'ensemble $\pi^{-1}(\pi(x))$ s'appelle *orbite critique* de f.

On dit qu'une application localement lipschitzienne Z-périodique $f : X \to \mathbf{R}$ satisfait la condition de Palais-Smale $(PS)_Z$ si, pour chaque suite (u_n) de X telle que $(f(u_n))$ est bornée et $\min_{x^* \in \partial f(u_n)} ||x^*|| \to 0$, la suite $(\pi(u_n))$ est relativement compacte dans X/Z.

Notre résultat abstrait est

THÉORÈME 19 *Soit $f : X \to \mathbf{R}$ une application localement Lipschitz, Z-périodique, bornée inférieurement et qui satisfait la condition de Palais-Smale $(PS)_Z$.*

Alors f a au moins $n+1$ orbites critiques distinctes, où n est la dimension de l'espace vectoriel engendré par Z.

Comme application de ce théorème on résout le problème multivoque du pendule forcé

$$\left\{ \begin{array}{l} x''(t) + f(t) \in [\underline{g}(x(t)), \overline{g}(x(t))] \quad \text{p.p. } t \in (0,1) \\ x(0) = x(1)\,, \end{array} \right. \tag{3.11}$$

où

$$f \in L^p(0,1) \quad \text{pour } \ p > 1 \tag{3.12}$$

$$g \in L^\infty(\mathbf{R}),\ g(u+T) = g(u) \quad \text{où } \ T > 0,\ \textit{p.p.} \ \ u \in \mathbf{R} \tag{3.13}$$

$$\int_0^T g(t)dt = \int_0^1 f(t)dt = 0\,. \tag{3.14}$$

On a noté

$$\underline{g}(u) = \lim_{\varepsilon \searrow 0} \text{essinf}\{g(u);\ |u-v| < \varepsilon\} \quad \overline{g}(u) = \lim_{\varepsilon \searrow 0} \text{esssup}\{g(u);\ |u-v| < \varepsilon\}\,.$$

Notre résultat d'existence est

THÉORÈME 20 *Supposons que les hypothèses (3.12)-(3.14) soient satisfaites. Alors le problème (3.11) a au moins deux solutions distinctes dans l'espace*

$$X := H^1_p(0,1) = \{x \in H^1(0,1);\ \ x(0) = x(1)\}\,.$$

3.2 Deux approches parallèles d'un problème non linéaire dans $\mathbf{R}^N$

En appliquant le théorème du col, Rabinowitz [42] a étudié le problème sans compacité

$$-\Delta u + b(x)u = f(x,u) \qquad \text{dans } \mathbf{R}^n\,,$$

où f est une fonction régulière, sous-critique et super-linéaire et $b \geq b_0 > 0$ dans $\mathbf{R}^n$. Notre but est de présenter deux variantes non lisses de ce problèmes, en utilisant les théories de point critique de Degiovanni et de Clarke.

Soit E l'espace de Hilbert des fonctions $u : \mathbf{R}^n \to \mathbf{R}$ telles que $\|u\|_E^2 := \int_{\mathbf{R}^n}(|Du|^2 + b(x)u^2) < \infty$. On suppose d'abord que l'opérateur linéaire $(-\Delta)$ est remplacé par un opérateur quasi-linéaire et on cherche les solutions faibles positives $u \in E$ du problème

$$(3.15)\quad -\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju + b(x)u = f(x,u) \quad x \in \mathbf{R}^n.$$

On impose les conditions suivantes

$$(3.16)\qquad \begin{cases} a_{ij} \equiv a_{ji} \\ a_{ij}(x,\cdot) \in C^1(\mathbf{R}) \quad \text{p.p. } x \in \mathbf{R}^n \\ a_{ij}(x,u)s,\ \frac{\partial a_{ij}}{\partial s}(x,u)s \in L^\infty(\mathbf{R}^n \times \mathbf{R})\ ; \end{cases}$$

$$(3.17)\ \exists \nu > 0 \quad \text{tel que} \quad \sum_{i,j=1}^{n} a_{ij}(x,u)s\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{p.p. } x \in \mathbf{R}^n,\ \forall s \in \mathbf{R},\ \forall \xi \in \mathbf{R}^n$$

$$(3.18)\ \begin{cases} \exists \mu \in (2,2^*)\,, \quad \gamma \in (0,\mu-2) \quad \text{tel que} \\ 0 \leq s\displaystyle\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u)s\xi_i\xi_j \leq \gamma\sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \quad \text{p.p. } x \in \mathbf{R}^n,\ \forall (s,\xi) \in \mathbf{R}\times\mathbf{R}^n\,. \end{cases}$$

Soit $b \in L^\infty_{\text{loc}}(\mathbf{R}^n)$ satisfaisant

$$(3.19)\qquad \begin{cases} \exists \underline{b} > 0 \quad \text{tel que} \quad b(x) \geq \underline{b} \quad \text{p.p. } x \in \mathbf{R}^n \\ \operatorname{ess}\lim\limits_{|x|\to\infty} b(x) = +\infty\,. \end{cases}$$

On suppose que $f(x,s) \not\equiv 0$ et

$$(3.20)\qquad \begin{cases} f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R} \text{ est une fonction de Carathéodory} \\ f(x,0) = 0 \qquad \text{p.p. } x \in \mathbf{R}^n \\ 0 \leq \mu F(x,s) \leq sf(x,s) \qquad \forall s \geq 0 \text{ p.p. } x \in \mathbf{R}^n\,. \end{cases}$$

La fonction f a une croissance sous-critique, exprimée par la condition

$$(3.21) \qquad \begin{cases} \forall \varepsilon > 0 \quad \exists f_\varepsilon \in L^{\frac{2n}{n+2}}(\mathbf{R}^n) \text{ tel que} \\ |f(x,s)| \leq f_\varepsilon(x) + \varepsilon |s|^{\frac{n+2}{n-2}} \qquad \forall s \in \mathbf{R} \text{ et p.p. } x \in \mathbf{R}^n \ . \end{cases}$$

Pour chaque $\delta \in (2, 2^*)$ on définit $q(\delta) = \frac{2n}{2n+(2-n)\delta}$ et on suppose que

$$(3.22) \qquad \begin{cases} \exists C \geq 0\ , \quad \exists \delta \in (2, 2^*)\ , \quad \exists G \in L^{q(\delta)}(\mathbf{R}^n) \quad \text{tel que} \\ F(x,s) \leq G(x)|s|^\delta + C|s|^{2^*} \qquad \forall s \in \mathbf{R},\ \text{p.p. } x \in \mathbf{R}^n \ . \end{cases}$$

On peut considérer aussi le cas $\delta = 2$ mais dans cette situation il faut supposer que $\|G\|_{n/2}$ est assez petit.

Notre résultat est

Théorème 21 *Supposons que les conditions (3.16)-(3.22) soient satisfaites. Alors le problème (3.15) admet au moins une solution positive dans E.*

On considère ensuite le problème multivoque

$$(3.23) \qquad -\Delta u + b(x)u \in [\underline{f}(x,u), \overline{f}(x,u)] \qquad \text{dans } \mathbf{R}^n\ ,$$

où

$$\underline{f}(x,s) = \lim_{\varepsilon \searrow 0} \text{ essinf } \{f(x,t);\ |t-s| < \varepsilon\} \qquad \overline{f}(x,s) = \lim_{\varepsilon \searrow 0} \text{ esssup } \{f(x,t);\ |t-s| < \varepsilon\}\ .$$

On suppose que $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ est une fonction mesurable telle que

$$(3.24) \qquad |f(x,s)| \leq C(|s| + |s|^p) \quad \text{p.p. } (x,s) \in \mathbf{R}^n \times \mathbf{R}\ ,$$

où $C > 0$ et $1 < p \leq \frac{n+2}{n-2}$. On ne suppose pas que $f(x,\cdot)$ est continue, mais si on définit $F(x,s) = \int_0^s f(x,t)dt$, on observe que F est une fonction de Carathéodory qui est localement Lipschitz par rapport à la deuxième variable. On observe aussi que $\Psi(u) = \int_{\mathbf{R}^n} F(x,u)$ est localement Lipschitz sur E.

On impose aussi les conditions

$$(3.25) \qquad \lim_{\varepsilon \searrow 0} \text{ esssup } \left\{ \left| \frac{f(x,s)}{s} \right|;\ (x,s) \in \mathbf{R}^n \times (-\varepsilon, \varepsilon) \right\} = 0$$

et il existe $\mu > 2$ tel que

$$(3.26) \qquad 0 \leq \mu F(x,s) \leq s\underline{f}(x,s) \quad \text{p.p. } (x,s) \in \mathbf{R}^n \times [0, +\infty)\ .$$

Notre résultat dans ce cas est

Théorème 22 *Supposons que les conditions (3.19), (3.24)-(3.26) soient satisfaites. Alors le problème (3.23) a au moins une solution positive dans E.*

3.3 Perturbations d'un problème non linéaire aux valeurs propres symétrique

Soit $\Omega \subset \mathbf{R}^N$ un ouvert borné. Pour $r > 0$ fixé arbitrairement on considère le problème suivant: trouver $(u, \lambda) \in H_0^1(\Omega) \times \mathbf{R}$ tel que

$$(3.27) \qquad \left\{ \begin{array}{l} f(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda f(x,u) \quad \text{dans } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,. \end{array} \right.$$

On suppose que $f : \Omega \times \mathbf{R} \to \mathbf{R}$ est une fonction de Carathéodory avec les propriétés suivantes:

$(f1)$ $f(x,-s) = -f(x,s)$, p.p. sur Ω et pour chaque $s \in \mathbf{R}$;

$(f2)$ il existe $a \in L^1(\Omega)$, $b \in \mathbf{R}$ et $0 \leq p < \frac{2N}{N-2}$ (si $N > 2$) tels que

$$0 < sf(x,s) \leq a(x) + b|s|^p\,, \qquad F(x,s) \leq a(x) + b|s|^p\,,$$

p.p. sur Ω et pour chaque $s \in \mathbf{R} \setminus \{0\}$, où $F(x,s) = \int_0^s f(x,t)dt$;

$(f3)$ $\sup\limits_{|s| \leq t} |f(x,s)| \in L^1_{loc}(\Omega)$, pour chaque $t > 0$.

Théorème 23 *Supposons que les conditions $(f1) - (f3)$ soient satisfaites. Alors le problème (3.27) admet une suite $(\pm u_n, \lambda_n)$ de solutions distinctes.*

Ensuite notre objectif est d'analyser le problème perturbé

$$(3.28) \qquad \left\{ \begin{array}{l} f(x,u),\, g(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda \left(f(x,u) + g(x,u) \right) \quad \text{dans } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,, \end{array} \right.$$

où $g : \Omega \times \mathbf{R} \to \mathbf{R}$ est une fonction de Carathéodory qui n'est pas nécessairement impaire par rapport à la seconde variable. On suppose quand même que g satisfait

$(g1)$ $0 < sg(x,s) \leq a(x) + b|s|^p$ p.p. sur Ω et pour chaque $s \in \mathbf{R} \setminus \{0\}$;

$(g2)$ $\sup\limits_{|s| \leq t} |g(x,s)| \in L^1_{loc}(\Omega)$, pour chaque $t > 0$;

$(g3)$ $G(x,s) \leq C_g\,(1 + |s|^p)$, p.p. sur Ω et pour chaque $s \in \mathbf{R}$, avec $C_g > 0$, où $G(x,s) = \int_0^s g(x,t)dt$.

On démontre que le nombre de solutions du problème perturbé (3.28) devient de plus en plus grand si la perturbation est assez petite, dans un sens précisé ultérieurement. Plus précisement, on a

Théorème 24 *Supposons que les conditions $(f1)-(f3)$ et $(g1)-(g3)$ soient satisfaites. Alors, pour chaque entier $n \geq 1$, il existe $\varepsilon_n > 0$ tel que le problème (3.28) admet au moins n solutions distinctes si g est une fonction telle que la condition $(g3)$ soit satisfaite pour $C_g = \varepsilon_n$.*

3.4 Résultats de multiplicité pour une classe d'inégalités hemivariationnelles

Soit $n \geq 2$ et Ω un ouvert borné et régulier de $\mathbf{R}^n$. Soit $\Psi : \mathbf{R}^{nN} \to \mathbf{R}$ une fonction convexe et paire telle que $\Psi(0) = 0$, $\Psi(\xi) > 0$ pour $\xi \neq 0$. On suppose aussi qu'il existe $c > 0$ tel que $\Psi(\xi) \leq c|\xi|$, pour chaque $\xi \in \mathbf{R}^{nN}$.

Soit $G : \Omega \times \mathbf{R}^N \to \mathbf{R}$ une application qui vérifie les conditions

(G_1) $G(\cdot, s)$ est mesurable, pour chaque $s \in \mathbf{R}^N$;

(G_2) pour chaque $t > 0$, il existe $\alpha_t \in L^1(\Omega)$ tel que

$$|G(x, s_1) - G(x, s_2)| \leq \alpha_t(x)|s_1 - s_2|$$

pour p.p. $x \in \Omega$ et chaque $s_1, s_2 \in \mathbf{R}^N$ avec $|s_j| \leq t$;

(G_3) il existe $a \in L^1(\Omega)$ et $b \in \mathbf{R}$ tels que

$$|G(x, s)| \leq a(x) + b|s|^p \qquad \text{pour p.p. } x \in \Omega \text{ et chaque } s \in \mathbf{R}^N\,;$$

(G_4) pour chaque $\varepsilon > 0$ il existe $a_\varepsilon \in L^1(\Omega)$ telle que

$$G^\circ(x, s; -s) \leq a_\varepsilon(x) + \varepsilon|s|^{n/(n-1)} \qquad \text{pour p.p. } x \in \Omega \text{ et chaque } s \in \mathbf{R}^N\,.$$

On suppose aussi que la fonction G satisfait les conditions

$$\text{(3.29)} \qquad \begin{cases} \text{il existe } \tilde{a} \in L^1(\Omega) \text{ et } \tilde{b} \in L^n(\Omega) \text{ tels que} \\ G(x, s) \geq -\tilde{a}(x) - \tilde{b}(x)|s| \qquad \text{pour p.p. } x \in \Omega \text{ et chaque } s \in \mathbf{R}^N\,; \end{cases}$$

$$\text{(3.30)} \qquad \lim_{|s| \to \infty} \frac{G(x, s)}{|s|} = +\infty \qquad \text{pour p.p. } x \in \Omega\,;$$

$$\text{(3.31)} \qquad \{s \longmapsto G(x, s)\} \text{ est paire pour p.p. } x \in \Omega\,.$$

Soit $\mathcal{E} : L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ définie par

$$\mathcal{E}(u) = \begin{cases} \displaystyle\int_\Omega \Psi(Du^a)\, dx + \int_\Omega \Psi^\infty\left(\frac{Du^s}{|Du^s|}\right) d|Du^s|(x) + \\ \displaystyle + \int_{\partial\Omega} \Psi^\infty(u \otimes \nu)\, d\mathcal{H}^{n-1}(x) & \text{si } u \in BV(\Omega; \mathbf{R}^N), \\ +\infty & \text{si } u \in L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N) \backslash BV(\Omega; \mathbf{R}^N), \end{cases}$$

où $Du = Du^a\, dx + Du^s$ est la décomposition de Lebesgue de Du, $|Du^s|$ est la variation totale de Du^s, $Du^s/|Du^s|$ est la dérivée de Radon-Nikodym de Du^s par rapport à $|Du^s|$, Ψ^∞ est la fonctionnelle de récession associée à Ψ, ν est la normale extérieure sur $\partial\Omega$ et la trace de u sur $\partial\Omega$ est encore designée par u (voir [1]).

Le théorème suivant est un résultat de multiplicité du type Clark [21].

THÉORÈME 25 *Pour chaque entier $k \geq 1$ il existe Λ_k tel que si $\lambda \geq \Lambda_k$, le problème*

$$\begin{cases} u \in BV(\Omega; \mathbf{R}^N) \\ \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x, u; v-u)\, dx \geq \lambda \int_\Omega \frac{u}{\sqrt{1+|u|^2}} \cdot (v-u)\, dx \quad \forall v \in BV(\Omega; \mathbf{R}^N) \end{cases}$$

admet au moins k paires $(u, -u)$ de solutions distinctes.

On impose maintenant la condition technique suplémentaire

$$\text{(3.32)} \quad \begin{cases} \text{il existe } q > 1 \text{ et } R > 0 \text{ tels que} \\ G^\circ(x, s; s) \leq qG(x, s) < 0 \ \text{ p.p. } x \in \Omega \text{ et chaque } s \in \mathbf{R}^N \text{ avec } |s| \geq R\,. \end{cases}$$

On définit la fonctionnelle paire et semicontinue inférieurement $f : L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ par

$$f(u) = \mathcal{E}(u) + \int_\Omega G(x, u)\, dx\,.$$

Sous ces hypothèses on montre

THÉORÈME 26 *Il existe une suite (u_h) de solutions du problème*

$$\begin{cases} u \in BV(\Omega; \mathbf{R}^N) \\ \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x, u; v-u)\, dx \geq 0 \quad \forall v \in BV(\Omega; \mathbf{R}^N) \end{cases}$$

telle que $f(u_h) \to +\infty$.

3.5 Un problème non linéaire qui modélise l'initiation des tremblements de terre

Soit $\Omega \subset \mathbf{R}^2$ un domaine régulier tel que sa frontière se décompose en deux parties: la frontière extérieure $\Gamma_d = \partial\Omega$ et une partie interne Γ composée d'un nombre fini d'arcs bornés. Soit

$$V = \{v \in H^1(\Omega);\ v = 0 \ \text{ sur } \ \Gamma_d\}$$

et soit $\gamma : V \to L^2(\Gamma)$ l'opérateur de trace. On définit le cône convexe fermé

$$K = \{v \in V;\ [v] \geq 0 \text{ sur } \Gamma\},$$

où $[\,\cdot\,]$ signifie le saut à travers Γ. Soit β une constante positive et $j(t) = -\beta t^2/2$, $t \in \mathbf{R}$. Pour $r > 0$ fixé on définit

$$M = \left\{u \in V;\ \int_\Omega u^2 dx = r^2\right\}.$$

On étudie le problème suivant de valeurs propres:

$$\text{(3.33)} \quad \begin{cases} \text{trouver } u \in K \cap M \text{ et } \lambda^2 \in \mathbf{R} \text{ tels que} \\ \int_\Omega \nabla u \cdot \nabla(v-u) dx + \int_\Gamma j'\left(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\right) d\sigma + \\ \lambda^2 \int_\Omega u(v-u) dx \geq 0, \quad \forall v \in K. \end{cases}$$

On montre le résultat suivant de multiplicité:

THÉORÈME 27 *Le problème (3.33) admet une infinité de solutions* (u, λ^2) *et l'ensemble de valeurs propres* $\{\lambda^2\}$ *satisfait* $\lambda_0^2 := \sup \lambda^2 < +\infty$ *et* $\inf \lambda^2 = -\infty$. *De plus, il existe une solution* (u_0, λ_0^2) *du problème (3.33). L'application* $\beta \longmapsto \lambda_0^2(\beta)$ *est convexe et*

$$\int_\Omega |\nabla v|^2 \, dx + \lambda_0^2(\beta) \int_\Omega v^2 \, dx \geq \beta \int_\Gamma [v]^2 \, d\sigma, \qquad \forall v \in K.$$

On étudie ensuite l'effet d'une perturbation arbitraire dans le problème (3.33). Plus précisèment, si ε est un réel, on considère le problème

$$(3.34) \quad \left\{ \begin{array}{l} \text{trouver } u_\varepsilon \in K \text{ et } \lambda_\varepsilon^2 \in \mathbf{R} \text{ tels que} \\ \int_\Omega \nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon) dx + \int_\Gamma (j' + \varepsilon g') \left(\gamma(u_\varepsilon(x)); \gamma(v(x)) - \gamma(u_\varepsilon(x))\right) d\sigma + \\ \lambda_\varepsilon^2 \int_\Omega u_\varepsilon (v - u_\varepsilon) dx \geq 0, \quad \forall v \in K, \end{array} \right.$$

où $\varepsilon > 0$ et $g : \mathbf{R} \to \mathbf{R}$ est une function continue arbitraire telle que

$$\begin{array}{ll} \exists\, a > 0,\ \exists\, 2 \leq p \leq \dfrac{2(N-1)}{N-2} \text{ tels que } |g(t)| \leq a(1 + |t|^p) & , \text{ si } N \geq 3; \\ \exists\, a > 0,\ \exists\, 2 \leq p < +\infty \text{ tels que } |g(t)| \leq a(1 + |t|^p) & , \text{ si } N = 2. \end{array}$$

On démontre que le nombre de solution du problème (3.34) devient de plus en plus grand si la perturbation "tend" vers zero:

THÉORÈME 28 *Pour chaque entier* $n \geq 1$, *il existe* $\varepsilon_n > 0$ *tel que le problème (3.34) a au moins* n *solutions distinctes* $(u_\varepsilon, \lambda_\varepsilon^2)$ *si* $|\varepsilon| < \varepsilon_n$. *De plus,* $\lambda_{0\varepsilon}^2 := \sup\{\lambda_\varepsilon^2\}$ *est fini et il existe une solution* $(u_{0\varepsilon}, \lambda_{0\varepsilon}^2)$ *du problème (3.34).*

4 Étude des inégalités hemivariationnelles

4.1 Perturbation d'une inégalité hemivariationnelle symétrique avec contrainte

Soit Ω un ouvert borné et régulier de $\mathbf{R}^N$ et soient $a_1, a_2 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}$ deux formes bilinéaires, symétriques et continues. Soient $B_1, B_2 : H_0^1(\Omega) \to H_0^1(\Omega)$ opérateurs linéaires, auto-adjoints et coercifs. Pour $a, b, r > 0$, soit

$$S_r^{a,b} = \{(v_1, v_2) \in H_0^1(\Omega) \times H_0^1(\Omega) \ : \ a(B_1 v_1, v_1) + b(B_2 v_2, v_2) = r^2\}.$$

Soit $j : \Omega \times \mathbf{R}^N \to \mathbf{R}$ telle que $j(x, \cdot)$ est localement Lipschitz. L'hypothèse fondamentale sur j est $j(x, -y) = j(x, y)$, pour p.p. $x \in \Omega$ et chaque $y \in \mathbf{R}^N$. On suppose que

(H_1) Il existe $\theta \in L^{\frac{p}{p-1}}(\Omega)$ et $\rho \in \mathbf{R}$ tels que $|z| \leq \theta(x) + \rho |y|^{p-1}$, pour p.p. $(x, y) \in \Omega \times \mathbf{R}^N$ et chaque $z \in \partial_y j(x, y)$.

On définit l'application $(A_1, A_2) : H_0^1(\Omega) \times H_0^1(\Omega) \to H^{-1}(\Omega) \times H^{-1}(\Omega)$ par la relation

$$\langle (A_1, A_2)(u_1, u_2), (v_1, v_2) \rangle = a_1(u_1, v_1) + a_2(u_2, v_2)$$

ainsi que l'application de dualité $\Lambda : H_0^1(\Omega) \times H_0^1(\Omega) \to H^{-1}(\Omega) \times H^{-1}(\Omega)$ par

$$\langle \Lambda(u_1, u_2), (v_1, v_2) \rangle = a(B_1 u_1, v_1) + b(B_2 u_2, v_2).$$

On impose la condition de compacité

(H_2) Pour chaque suite $\{(u_n^1, u_n^2)\} \subset S_r^{a,b}$ telle que $u_n^i \rightharpoonup u_i$ dans $H_0^1(\Omega)$, pour chaque $z_n^i \in \partial f_i(u_n^i)$, tel que

$$a_i(u_n^i, u_n^i) + \langle z_n^i, u_n^i \rangle_V \to \alpha_i \in \mathbf{R}, \tag{4.35}$$

$i = 1, 2$, et pour chaque $w \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ satisfaisant

$$w(x) \in \partial_y j(x, (u_1 - u_2)(x)) \text{ pour p.p. } x \in \Omega, \tag{4.36}$$

tel que

$$[(A_1, A_2) - \lambda_0 \cdot \Lambda]\,(u_n^1, u_n^2)$$

converge dans $H^{-1}(\Omega) \times H^{-1}(\Omega)$, où

$$\lambda_0 = r^{-2}(\alpha_1 + \alpha_2 + \int_\Omega \langle w(x), (u_1 - u_2)(x) \rangle\, dx), \tag{4.37}$$

il existe une sous-suite convergente de (u_n^1, u_n^2).

Soit $g : \Omega \times \mathbf{R}^N \to \mathbf{R}$ une fonction de Carathéodory qui est localement Lipschitz par rapport à la deuxième variable et qui ne satisfait aucune hypothèse de parité. On impose

(H_3) Il existe $\theta_1 \in L^{p/(p-1)}(\Omega)$ et $\theta_2 \in L^\infty(\Omega)$ tels que

$$|z| \le \theta_1(x) + \theta_2(x)|y|^{p-1},$$

pour p.p. $(x, y) \in \Omega \times \mathbf{R}^N$ et chaque $z \in \partial_y g(x, y)$.

Pour $\phi \in H^{-1}(\Omega)$, on considère le problème suivant: trouver $(u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ et $(\lambda_1, \lambda_2) \in \mathbf{R}^2$ tels que

$$(P_{r,a,b}) \begin{cases} a_1(u_1, v_1) + a_2(u_2, v_2) + < \phi, v_1 > + < \phi, v_2 > + \\ + \int_\Omega \{ j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x)) + \\ g_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x)) \} dx \ge \\ \ge \lambda_1 (B_1 u_1, v_1)_V + \lambda_2 (B_2 u_2, v_2)_V, \ \forall\, v_1, v_2 \in H_0^1(\Omega), \\ \\ a(B_1 u_1, u_1) + b(B_2 u_2, u_2) = r^2. \end{cases}$$

Motreanu et Panagiotopoulos [36] ont montré que si $g = 0$ et $\phi = 0$, alors le problème $(P_{r,a,b})$ admet une infinité de solutions. Notre résultat est dans le même esprit que le théorème 23 et montre que si les perturbations g et ϕ sont assez petites, alors le nombre de solutions de ce problème devient de plus en plus grand. Plus précisément, on a

THÉORÈME 29 *Supposons que les hypothèses $(H_1) - (H_3)$ soient satisfaites. Alors, pour chaque $n \ge 1$, il existe $\delta_n > 0$ tel que, si $||\phi||_{H^{-1}} \le \delta_n$ et si $||\theta_1||_{L^{p/(p-1)}} \le \delta_n$, $||\theta_2||_{L^\infty} \le \delta_n$, alors le problème $(P_{r,a,b})$ a au moins n solutions distinctes.*

4.2 Résultats d'existence du type Hartman-Stampacchia pour les inégalités hemivariationnelles

Soit V un espace de Banach reflexif infini dimensionnel tel qu'il existe $T : V \to L^p(\Omega, \mathbf{R}^k)$ un opérateur linéaire et continu, où $1 \leq p < \infty$, $k \geq 1$, et Ω est un ouvert borné de $\mathbf{R}^N$. Soit $K \subset V$, $A : K \to V^*$ et $j = j(x,y) : \Omega \times \mathbf{R}^k \to \mathbf{R}$ une fonction de Carathéodory qui est localement Lipschitz par rapport à la deuxième variable $y \in \mathbf{R}^k$ et qui satisfait la condition

(j) il existe $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbf{R})$ et $h_2 \in L^\infty(\Omega, \mathbf{R})$ tels que

$$|z| \leq h_1(x) + h_2(x)|y|^{p-1},$$

pour p.p. $x \in \Omega$ et chaque $y \in \mathbf{R}^k$, $z \in \partial j(x,y)$. Soit $Tu = \hat{u}$, $u \in V$. On étudie le problème

(P) Trouver $u \in K$ tel que, pour chaque $v \in K$,

$$\langle Au, v-u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0.$$

On montre plusieurs résultats d'existence pour ce problème, dont on cite

THÉORÈME 30 *Supposons que l'ensemble K est fermé, borné et convexe et que l'opérateur $A : K \to V^*$ est monotone et demi-continu sur $F \cap K$, pour chaque sous-espace fini dimensionel de V. Si j satisfait la condition (j) alors le problème (P) a au moins une solution.*

On montre aussi plusieurs résultats d'existence pour des inégalités variationnelles-hemivariationnelles du type: trouver $u \in K$ tel que

$$\langle Au - f, v-u\rangle + \Phi(v) - \Phi(u) + \int_T j^0(x, \gamma(u(x)); \gamma(v(x) - u(x)))d\mu \geq 0\,, \qquad \forall v \in K\,.$$

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Chapitre I
Problèmes elliptiques semi-linéaires et quasi-linéaires: existence et unicité des solutions

1. F. Cîrstea et V. Rădulescu, Existence and uniqueness of positive solutions to a semilinear elliptic problem in $\mathbf{R}^N$, *J. Math. Anal. Appl.* **229** (1999), 417-425.

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Existence and Uniqueness of Positive Solutions to a Semilinear Elliptic Problem in $\mathbf{R}^N$

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Let $p \in C^{0,\alpha}_{loc}(\mathbf{R}^N)$ with $p > 0$ and let $f \in C^1((0,\infty),(0,\infty))$ be such that $\lim_{u\searrow 0} f(u)/u = +\infty$, f is bounded at infinity and the mapping $u \longmapsto f(u)/(u+\beta)$ is decreasing on $(0,\infty)$, for some $\beta > 0$. We prove that the problem $-\Delta u = p(x)f(u)$ in $\mathbf{R}^N$, $N > 2$, has a unique positive $C^{2,\alpha}_{loc}(\mathbf{R}^N)$ solution which vanishes at infinity provided $\int_0^\infty r\Phi(r)dr < \infty$, where $\Phi(r) = \max\{p(x); |x| = r\}$. Furthermore, it is showed that this condition is nearly optimal. Our results extend previous works by Lair-Shaker and Zhang, while the proofs are based on two theorems on bounded domains, due to Brezis-Oswald and Crandall-Rabinowitz-Tartar.

1 Introduction

Consider the problem

$$\begin{cases} -\Delta u = p(x)f(u) \quad \text{in} \quad \mathbf{R}^N \\ u > 0 \quad \text{in} \quad \mathbf{R}^N \\ u(x) \to 0 \quad \text{as} \quad |x| \to \infty\,, \end{cases} \tag{1}$$

where $N > 2$ and the function p satisfies the following hypotheses:
(p1) $p \in C^{0,\alpha}_{\mathrm{loc}}(\mathbf{R}^N)$ for some $\alpha \in (0,1)$;
(p2) $p > 0$ in $\mathbf{R}^N$.

This problem has been intensively studied in the case where $f(u) = u^{-\gamma}$, with $\gamma > 0$. For instance, in the case of a bounded domain $\Omega \subset \mathbf{R}^N$, Lazer and McKenna proved in [7] that the problem

$$-\Delta u = p(x)u^{-\gamma}\,, \qquad \text{in } \Omega$$

has a unique classical solution if p is a sufficiently smooth function which is positive on $\overline{\Omega}$. The existence of entire positive solutions on $\mathbf{R}^N$ for $\gamma \in (0,1)$ and under certain additional hypotheses has been established in Edelson [4] and in Kusano-Swanson [5]. For instance, Edelson proved the existence of a solution provided that

$$\int_1^{\infty} r^{N-1+\lambda(N-2)}\Phi(r)dr < \infty\,,$$

for some $\lambda \in (0,1)$, where $\Phi(r) = \max_{|x|=r} p(x)$. This result is generalized for any $\gamma > 0$ via the sub and super solutions method in Shaker [8] or by other methods in Dalmasso [3]. Lair and Shaker continued in [6] the study of (1) for $f(u) = u^{-\gamma}$, $\gamma > 0$. They proved the existence of a solution under the hypothesis

(p3) $\int\limits_0^{\infty} r \cdot \Phi(r)dr < \infty$, where $\Phi(r) = \max_{|x|=r} p(x)$.

Zhang studied in [9] the case of a nonlinearity $f \in C^1((0,\infty),(0,\infty))$ which decreases on $(0,\infty)$ and satisfying $\lim_{u\searrow 0} f(u) = +\infty$.

Our aim is to extend the results of Lair, Shaker and Zhang for the case of a nonlinearity which is not necessarily decreasing on $(0,\infty)$. More exactly, let $f : (0,\infty) \to (0,\infty)$ be a C^1 function which satisfies the following assumptions:

(f1) there exists $\beta > 0$ such that the mapping $u \longmapsto \dfrac{f(u)}{u+\beta}$ is decreasing on $(0,\infty)$;

(f2) $\lim_{u\searrow 0} \dfrac{f(u)}{u} = +\infty$ and f is bounded in a neighbourhood of $+\infty$.

Our main result is the following:

Theorem 1 *Under hypotheses (f1), (f2), (p1)-(p3), the problem (1) has a unique positive global solution* $u \in C^{2,\alpha}_{\rm loc}(\mathbf{R}^N)$.

Theorem 1 shows that (p3) is sufficient for the existence of the unique solution to the problem (1). The following result shows that condition (p3) is nearly necessary.

Theorem 2 *Suppose p is a positive radial function which is continuous on $\mathbf{R}^N$ and satisfies*

$$\int\limits_0^{\infty} rp(r)dr = \infty.$$

Then the problem (1) has no positive radial solution.

2 Uniqueness

Suppose u and v are arbitrary solutions of the problem (1). Let us show that $u \le v$ or, equivalently, $\ln(u(x)+\beta) \le \ln(v(x)+\beta)$, for any $x \in \mathbf{R}^N$. Assume the contrary. Since we have

$$\lim_{|x|\to\infty} (\ln(u(x)+\beta) - \ln(v(x)+\beta)) = 0,$$

we deduce that $\max_{\mathbf{R}^N} (\ln(u(x)+\beta) - \ln(v(x)+\beta))$ exists and is positive. At that point, say x_0, we have

$$\nabla(\ln(u(x_0)+\beta) - \ln(v(x_0)+\beta)) = 0,$$

so

$$\frac{1}{u(x_0)+\beta}\cdot\nabla u(x_0)=\frac{1}{v(x_0)+\beta}\cdot\nabla v(x_0)\,. \tag{2}$$

By (f1) we obtain

$$\frac{f(u(x_0))}{u(x_0)+\beta}<\frac{f(v(x_0))}{v(x_0)+\beta}\,. \tag{3}$$

So, by (2) and (3),

$$0\geq\Delta\left(\ln(u(x_0)+\beta)-\ln(v(x_0)+\beta)\right)=\frac{1}{u(x_0)+\beta}\cdot\Delta u(x_0)-\frac{1}{v(x_0)+\beta}\cdot\Delta v(x_0)-$$
$$\frac{1}{(u(x_0)+\beta)^2}\cdot|\nabla u(x_0)|^2+\frac{1}{(v(x_0)+\beta)^2}\cdot|\nabla v(x_0)|^2=$$
$$\frac{1}{u(x_0)+\beta}\,\Delta u(x_0)-\frac{1}{v(x_0)+\beta}\,\Delta v(x_0)=$$
$$-p(x_0)\left(\frac{f(u(x_0))}{u(x_0)+\beta}-\frac{f(v(x_0))}{v(x_0)+\beta}\right)>0\,,$$

which is a contradiction. Hence $u\leq v$. A similar argument can be made to produce $v\leq u$, forcing $u=v$.

3 Existence

We first show that our hypothesis (f1) implies that $\lim\limits_{u\searrow 0}f(u)$ exists, finite or $+\infty$. Indeed, since $\frac{f(u)}{u+\beta}$ is decreasing, there exists $L:=\lim\limits_{u\searrow 0}\frac{f(u)}{u+\beta}\in(0,+\infty]$. It follows that $\lim\limits_{u\searrow 0}f(u)=L\beta$.

In order to prove the existence of a solution to (1), we need to employ a corresponding result by Brezis and Oswald (see [1]) for bounded domains. They considered the problem

$$\begin{cases}-\Delta u=g(x,u) & \text{in } \Omega\\ u\geq 0,\quad u\not\equiv 0 & \text{in } \Omega\\ \quad u=0 & \text{on } \partial\Omega\,,\end{cases} \tag{4}$$

where $\Omega\subset\mathbf{R}^N$ is a bounded domain with smooth boundary and $g(x,u):\Omega\times[0,\infty)\to\mathbf{R}$.

Assume that

$$\begin{cases}\text{for a.e. } x\in\Omega\ \text{ the function } u\to g(x,u)\ \text{ is continuous on }\ [0,\infty)\\ \text{and the function } u\to g(x,u)/u \text{ is decreasing on } (0,\infty)\,;\end{cases} \tag{5}$$

$$\text{for each } u\geq 0 \text{ the function } x\to g(x,u) \text{ belongs to } L^\infty(\Omega)\,; \tag{6}$$

$$\exists C > 0 \text{ such that } g(x,u) \leq C(u+1) \text{ a.e. } x \in \Omega, \quad \forall\, u \geq 0. \tag{7}$$

Set

$$a_0(x) = \lim_{u \searrow 0} g(x,u)/u \qquad \text{and} \qquad a_\infty(x) = \lim_{u \to \infty} g(x,u)/u\,,$$

so that $-\infty < a_0(x) \leq +\infty$ and $-\infty \leq a_\infty(x) < +\infty$.

Under these hypotheses on g, Brezis and Oswald proved in [1] that there is at most one solution of (4). Moreover, a solution of (4) exists if and only if

$$\lambda_1(-\Delta - a_0(x)) < 0 \tag{8}$$

and

$$\lambda_1(-\Delta - a_\infty(x)) > 0, \tag{9}$$

where $\lambda_1(-\Delta - a(x))$ denotes the first eigenvalue of the operator $-\Delta - a(x)$ with zero Dirichlet condition. The precise meaning of $\lambda_1(-\Delta - a(x))$ is

$$\lambda_1(-\Delta - a(x)) = \inf_{\varphi \in H_0^1,\ \|\varphi\|_2 = 1} \left(\int |\nabla\varphi|^2 - \int_{[\varphi \neq 0]} a\varphi^2 \right).$$

Note that $\displaystyle\int_{[\varphi\neq 0]} a\varphi^2$ makes sense if $a(x)$ is any measurable function such that either $a(x) \leq C$ or $a(x) \geq -C$ a.e. on Ω.

Let us consider the problem

$$\begin{cases} -\Delta u_k = p(x) f(u_k)\,, \text{ if } \mid x \mid < k \\ u_k(x) = 0\,, \quad \text{if } \mid x \mid = k\,. \end{cases} \tag{10}$$

The following two distinct situations may occur:

Case 1: f is bounded on $(0,\infty)$.

In this case, as we have initially observed, there exists and it is finite $\lim_{u \searrow 0} f(u)$, so f can be extended by continuity at the origin.

In order to obtain a solution to the problem (10), it is enough to verify that the hypotheses of the Brezis-Oswald theorem are fulfilled. Obviously, (5) and (6) hold. Now, using (p1), (p2) and the fact that f is bounded, we easily deduce that (7) is satisfied. We observe that $a_0(x) = \lim_{u \searrow 0} \dfrac{p(x)f(u)}{u} = +\infty$ and $a_\infty(x) = \lim_{u \to +\infty} \dfrac{p(x)f(u)}{u} = 0$. Then (8) and (9) are also fulfilled. Thus by Theorem 1 in [1] the problem (10) has a unique solution u_k which, by the maximum principle, is positive in $|x| < k$.

Case 2: $\lim_{u \searrow 0} f(u) = +\infty$.

We will apply the method of sub and supersolutions in order to find a solution to the problem (10). We first observe that 0 is a subsolution for this problem.

We construct in what follows a positive supersolution. By the boundedness of f in a neighbourhood of $+\infty$, there exists $A > 0$ such that $f(u) \leq A$, for any $u \in (1, +\infty)$. Let $f_0 : (0, 1] \to (0, +\infty)$ be a continuous nonincreasing function such that $f_0 \geq f$ on $(0, 1]$. We can assume without loss of generality that $f_0(1) = A$. Set

$$g(u) = \begin{cases} f_0(u)\,, \text{ if } 0 < u \leq 1 \\ A\,, \quad \text{if } u > 1\,. \end{cases}$$

Then g is a continuous nonincreasing function on $(0, +\infty)$. Let $h : (0, \infty) \to (0, \infty)$ be a C^1 nonincreasing function such that $h \geq g$. Thus, by Theorem 1.1 in [2] the problem

$$\begin{cases} -\Delta U = p(x)h(U) \quad \text{if } \mid x \mid < k \\ U = 0, \quad \text{if } \mid x \mid = k \end{cases}$$

has a positive solution. Now, since $h \geq f$ on $(0, +\infty)$, it follows that U is supersolution for the problem (10).

In both cases studied above we define $u_k = 0$ for $|x| > k$. Using a maximum principle argument as already done above for proving the uniqueness, we can show that $u_k \leq u_{k+1}$ on $\mathbf{R}^N$.

We now prove the existence of a positive function $v \in C^2(\mathbf{R}^N)$ for which $u_k \leq v$ on $\mathbf{R}^N$. As in [6] we construct first a positive radially symmetric function w such that $-\Delta w = \Phi(r)$ $(r = \mid x \mid)$ on $\mathbf{R}^N$ and $\lim_{r\to\infty} w(r) = 0$. We obtain

$$w(r) = K - \int_0^r \zeta^{1-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma) d\sigma d\zeta\,,$$

where

$$K = \int_0^\infty \zeta^{1-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\, d\sigma\, d\zeta\,, \tag{11}$$

provided the integral is finite. Integration by parts gives

$$\begin{gathered} \int_0^r \zeta^{1-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\, d\sigma\, d\zeta = -(n-2)^{-1} \int_0^r \frac{d}{d\zeta}\zeta^{2-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\, d\sigma\, d\zeta = \\ (n-2)^{-1} \left(-r^{2-n} \int_0^r \sigma^{n-1}\Phi(\sigma)\, d\sigma + \int_0^r \zeta\Phi(\zeta)\, d\zeta \right)\,. \end{gathered} \tag{12}$$

Now, using L'Hopital's rule, we evaluate the limit of the right side of (12) as $r \to \infty$. We have

$$\lim_{r\to\infty}\left(-r^{2-n}\int_0^r \sigma^{n-1}\Phi(\sigma)\,d\sigma + \int_0^r \zeta\Phi(\zeta)\,d\zeta\right) =$$

$$= \lim_{r\to\infty}\frac{-\int_0^r \sigma^{n-1}\Phi(\sigma)\,d\sigma + r^{n-2}\int_0^r \zeta\Phi(\zeta)\,d\zeta}{r^{n-2}} =$$

$$= \lim_{r\to\infty}\int_0^r \zeta\Phi(\zeta)\,d\zeta = \int_0^\infty \zeta\Phi(\zeta)\,d\zeta < \infty.$$

Then we obtain $K = \frac{1}{n-2}\cdot\int_0^\infty \zeta\Phi(\zeta)\,d\zeta < \infty$.

Clearly, we have

$$w(r) < \frac{1}{n-2}\cdot\int_0^\infty \zeta\Phi(\zeta)d\zeta \quad \forall r > 0.$$

Let v be a positive function such that $w(r) = \frac{1}{c}\cdot\int_0^{v(r)}\frac{t}{f(t)}dt$, where $c > 0$ will be chosen such that $Kc \le \int_0^c \frac{t}{f(t)}dt$.

We prove that we can find $c > 0$ with this property.

By our hypothesis (f2) we obtain that $\lim_{x\to\infty}\int_0^x \frac{t}{f(t)}dt = +\infty$. Now using L'Hopital's rule we have

$$\lim_{x\to\infty}\frac{\int_0^x \frac{t}{f(t)}dt}{x} = \lim_{x\to\infty}\frac{x}{f(x)} = +\infty\,.$$

From this we deduce that there exists $x_1 > 0$ such that $\int_0^x \frac{t}{f(t)}dt \ge Kx$ for all $x \ge x_1$. It follows that for any $c \ge x_1$ we have $Kc \le \int_0^c \frac{t}{f(t)}dt$.

But w is a decreasing function, and this implies that v is a decreasing function too. Then $\int_0^{v(r)}\frac{t}{f(t)}dt \le \int_0^{v(0)}\frac{t}{f(t)}dt = c\cdot w(0) = c\cdot K \le \int_0^c \frac{t}{f(t)}dt$.
It follows that $v(r) \le c$ for all $r > 0$.

From $w(r) \to 0$ as $r \to \infty$ we deduce that $v(r) \to 0$ as $r \to \infty$.

By the choice of v we have

$$\nabla w = \frac{1}{c} \cdot \frac{v}{f(v)} \nabla v \quad \text{and} \quad \Delta w = \frac{1}{c} \frac{v}{f(v)} \Delta v + \frac{1}{c} \left(\frac{v}{f(v)} \right)' |\nabla v|^2. \tag{13}$$

The hypothesis $u \longmapsto \dfrac{f(u)}{u+\beta}$ is a decreasing function on $(0, \infty)$ implies that $u \longmapsto \dfrac{f(u)}{u}$ is a decreasing function on $(0, \infty)$. From (13) we deduce that

$$\Delta v < c \frac{f(v)}{v} \Delta w = -c \frac{f(v)}{v} \Phi(r) \leq -f(v) \Phi(r) \,. \tag{14}$$

By (10) and (14) and using in an essential manner the hypothesis (f1), as already done for proving the uniqueness, we obtain that $u_k \leq v$ for $|x| \leq k$ and, hence, for all $\mathbf{R}^N$.

Now we have a bounded increasing sequence

$$u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq v \,,$$

with v vanishing at infinity. Thus there exists a function, say $u \leq v$ such that $u_k \to u$ pointwise in $\mathbf{R}^N$.

Now, using the same argument as in [6], it is easy to prove that $u \in C^{2,\alpha}_{loc}(\mathbf{R}^N)$ and thus u is a classical solution of the problem (1).

4 Proof of Theorem 2

Suppose (1) has such a solution, $u(r)$. Then

$$u''(r) + \frac{n-1}{r} u'(r) = -f(u(r))p(r).$$

We put $\ln(u(r)+1) = \tilde{u}(r) > 0$ for all $r > 0$.

$$\Delta \tilde{u}(r) = \frac{1}{u(r)+1} \Delta u(r) - \frac{1}{(u(r)+1)^2} |\nabla u|^2.$$

Then $\tilde{u}(r)$ satisfies

$$\tilde{u}'' + \frac{n-1}{r} \tilde{u}' + \frac{1}{(u(r)+1)^2} |\nabla u|^2 = -\frac{f(u(r))}{u(r)+1} p(r). \tag{15}$$

Multiplying equation (15) by r^{n-1} and integrating on $(0, \zeta)$ yield

$$\tilde{u}'(\zeta)\zeta^{n-1} + \int_0^\zeta \frac{\sigma^{n-1}}{(u(\sigma)+1)^2} |\nabla u|^2 d\sigma = -\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1} p(\sigma) \sigma^{n-1} d\sigma. \tag{16}$$

Now we multiply (16) by ζ^{1-n} and integrate over $(0,r)$. Hence

$$\tilde{u}(r)-\tilde{u}(0)+\int\limits_0^r \zeta^{1-n}\int\limits_0^\zeta \frac{\sigma^{n-1}}{(u(\sigma)+1)^2}|\nabla u|^2 d\sigma\, d\zeta =$$

$$=-\int\limits_0^r \zeta^{1-n}\int\limits_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta.$$

We observe that $\tilde{u}(r)<\tilde{u}(0)\ \forall r>0$ implies $u(r)<u(0)\ \forall r>0$.

If $\beta\geq 1$ then the function $u\longmapsto \dfrac{f(u)}{u+1}$ is decreasing on $(0,\infty)$. This implies

$$\frac{f(u(\sigma))}{u(\sigma)+1}>\frac{f(u(0))}{u(0)+1}. \tag{17}$$

Since $\tilde{u}$ is positive, we have

$$\int\limits_0^r \zeta^{1-n}\int\limits_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta \leq \tilde{u}(0) \quad \text{for all } r>0\,.$$

Substituting (17) into this expression we obtain

$$\int\limits_0^r \zeta^{1-n}\int\limits_0^\zeta p(\sigma)\sigma^{n-1}d\sigma\, d\zeta \leq \frac{u(0)+1}{f(u(0))}\tilde{u}(0)<\infty.$$

We can use integration by parts and L'Hopital's rule (as we did in proving that the integral in (11) is finite) to rewrite this as

$$\frac{1}{n-2}\lim_{r\to\infty}\int\limits_0^r tp(t)dt \leq \frac{u(0)+1}{f(u(0))}\tilde{u}(0)<\infty\,.$$

contradicting the hypothesis.

If $\beta<1$ then the function $u\longmapsto \dfrac{u+\beta}{u+1}$ is increasing on $(0,\infty)$. In this case we have

$$\tilde{u}(0)>\int\limits_0^r \zeta^{1-n}\int\limits_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta =$$

$$=\int\limits_0^r \zeta^{1-n}\int\limits_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+\beta}\cdot\frac{u(\sigma)+\beta}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta \geq$$

$$\geq \frac{f(u(0))}{u(0)+\beta}\beta \int_0^r \zeta^{1-n} \int_0^\zeta p(\sigma)\sigma^{n-1} d\sigma\, d\zeta$$

which implies

$$\int_0^r \zeta^{1-n} \int_0^\zeta p(\sigma)\sigma^{n-1} d\sigma\, d\zeta < \frac{\tilde{u}(0)(u(0)+\beta)}{\beta \cdot f(u(0))} < \infty \quad \text{for all } r > 0.$$

We obtain again that

$$\frac{1}{n-2} \lim_{r\to\infty} \int_0^r tp(t)dt \leq \frac{u(0)+\beta}{\beta \cdot f(u(0))} \tilde{u}(0) < \infty$$

contradicting the hypothesis.

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Multiple solutions of degenerate perturbed elliptic problems involving a subcritical Sobolev exponent

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This paper is dedicated with admiration to Professor H. Brezis on the occasion of his 55th birthday anniversary

Abstract. We study the degenerate elliptic equation

$$-\mathrm{div}\,(a(x)\nabla u)+b(x)\,u=K(x)|u|^{p-2}u+g(x)\quad \text{in } \mathbf{R}^N,$$

where $N\geq 2$ and $2<p<2^*$. We assume that $a\not\equiv 0$ is a continuous, bounded and nonnegative function, while b and K are positive and essentially bounded in $\mathbf{R}^N$. Under some assumptions on a, b and K, which control the location of zeros of a and the behaviour of a, b and K at infinity we prove that if the perturbation g is sufficiently small then the above problem has at least two distinct solutions in an appropriate weighted Sobolev space. The proof relies essentially on the Ekeland Variational Principle [8] and on the Mountain Pass Theorem without the Palais-Smale condition, established in Brezis-Nirenberg [6], combined with a weighted variant of the Brezis-Lieb Lemma [5], in order to overcome the lack of compactness .

Key words: degenerate elliptic problem, weighted Sobolev space, unbounded domain, perturbation, multiple solutions.

1 Introduction

Perturbations of semilinear elliptic equations and of inequality value problems have been intensively studied in the last two decades. We start with the elementary example

$$\begin{cases} -\Delta u=|u|^{p-2}u & \text{in } \Omega\,,\\ u=0 & \text{on } \partial\Omega\,, \end{cases}\tag{1}$$

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where Ω is a smooth bounded domain in $\mathbf{R}^N$ $(N \geq 2)$ and $2 < p < 2^*$. Here 2^* denotes the critical Sobolev exponent, that is, $2^* = 2N/(N-2)$, if $N \geq 3$, and $2^* = +\infty$, if $N = 2$. A classical result, based on a $\mathbf{Z}_2$ symmetric version of the Mountain Pass Theorem (see Ambrosetti-Rabinowitz [1]), implies that problem (1) has infinitely many solutions in $H_0^1(\Omega)$. A natural question is to see what happens if the above problem is affected by a certain perturbation. Consider the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u + g(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2}$$

Bahri-Berestycki [3] and Struwe [14] have showed independently that there exists $p_0 < 2^*$ such that for any $g \in L^2(\Omega)$, problem (2) still has infinitely many solutions, provided $2 < p < p_0$. Moreover, Bahri [2] has shown that for any $2 < p < p_0$ there is a dense open set of $g \in H^{-1}(\Omega)$ for which problem (2) possesses infinitely many solutions.

Our aim is to study a perturbation problem, but from another point of vue. More exactly, we will analyse the effect of a small perturbation g in the degenerate semilinear elliptic problem

$$-\mathrm{div}\,(a(x)\nabla u) + b(x)\,u = K(x)|u|^{p-2}u + g(x) \quad \text{in } \mathbf{R}^N, \tag{3}$$

where $N \geq 2$ and $2 < p < 2^*$. Suppose that $a \in C(\mathbf{R}^N)$ and $b,\ K \in L^\infty(\mathbf{R}^N)$ satisfy the hypotheses:

$(A1)$ There exists $R_0 > 0$ such that

$$\{x;\, a(x) = 0\} \subset B(0, R_0) \quad \text{and} \quad \frac{1}{a} \in L^q(B(0, R_0)) \text{ for some } q > \frac{Np}{2N + 2p - Np}\,;$$

$(A2)$ $\lim\limits_{|x|\to\infty} a(x) = a(\infty) \in \mathbf{R}_+$ and $0 \leq a(x) \leq a(\infty)$ in $\mathbf{R}^N$;

(B) $\operatorname{ess}\lim\limits_{|x|\to\infty} b(x) = b(\infty) \in \mathbf{R}_+$ and there exists $b_1 > 0$ such that $b_1 \leq b(x) \leq b(\infty)$ a.e. in $\mathbf{R}^N$;

(K) $\operatorname{ess}\lim\limits_{|x|\to\infty} K(x) = K(\infty) \in \mathbf{R}_+$ and $K(x) \geq K(\infty)$ a.e. in $\mathbf{R}^N$;

(M) $\operatorname{meas}(\{x \in \mathbf{R}^N;\, b(x) < b(\infty)\} \cup \{x \in \mathbf{R}^N;\, K(x) > K(\infty)\}) > 0\,.$

The degeneracy hypothesis $(A1)$ is inspired by condition $(A-1)$ introduced in Murthy-Stampacchia [11]. In light of Proposition 1, assumption $(A1)$ should be seen as a "subcritically" condition. Our framework includes degeneracies a that behave like $a(x) \sim |x|^\alpha$ near the origin, with $0 < \alpha < 2N/p + 2 - N$. For the treatment of supercritical degeneracies on bounded domains we refer to Passaseo [12], where several nonexistence results are proven. The full strength of condition $(A2)$ will appear in the proof of Proposition 2. This assumption is taken over from Chabrowski [7] and it will be used in this paper only to check that the hypotheses of [7, Theorem 1] are fulfilled in our situation.

Let $H^1_{a,b}(\mathbf{R}^N)$ be the Sobolev space defined as the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the norm

$$\|u\|_{a,b}^2 = \int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\, dx.$$

We denote by $\|\cdot\|_{-1}$ the norm of $H_{a,b}^{-1}(\mathbf{R}^N)$ which is the dual space of $H_{a,b}^1(\mathbf{R}^N)$, i. e. $H_{a,b}^{-1}(\mathbf{R}^N) = (H_{a,b}^1(\mathbf{R}^N))^*$. Throughout this work we suppose that $g \in H_{a,b}^{-1}(\mathbf{R}^N) \setminus \{0\}$.

Definition 1 *We say that $u \in H_{a,b}^1(\mathbf{R}^N)$ is a weak solution of (3) if*

$$\int_{\mathbf{R}^N} (a(x)\nabla u \cdot \nabla v + b(x)uv)\, dx - \int_{\mathbf{R}^N} K(x)|u|^{p-2}uv\, dx - \int_{\mathbf{R}^N} g(x)v\, dx = 0 \quad \textit{for all } v \in C_0^\infty(\mathbf{R}^N)\,.$$

We are concerned in this paper with the study of the degenerate semilinear elliptic equation (3), in other words it is assumed that a vanishes in at least one point in $\mathbf{R}^N$. The main result asserts that if $\|g\|_{-1}$ is sufficiently small then problem (3) possesses at least two solutions. We overcome the lack of compactness of our problem by applying a variant of the Mountain Pass Theorem without the Palais-Smale condition (see Brezis-Nirenberg [6, Theorem 2.2]), combined with a generalization of the Brezis-Lieb Lemma [5, Theorem 1]. We also point out that the study of degenerate elliptic boundary value problems was initiated in Mikhlin [9], [10] and many papers have been devoted in the past decades to the study of several questions related to these problems. We refer only to Murthy-Stampacchia [11], Stredulinsky [13], Passaseo [12] and the references therein.

Taking into account our hypothesis $(A2)$, the continuity of a implies that meas $\{x \in \mathbf{R}^N;\ a(x) < a(\infty)\} > 0$. On the other hand, combining the hypotheses $(A1)$ and $(A2)$ with the continuity of a we obtain that $\inf_{\mathbf{R}^N \setminus B(0,R_0)} a(x) > 0$. According to these comments we see that if a, $K \in C(\mathbf{R}^N)$ satisfy $(A1)$, $(A2)$ and (K) then all the assumptions of Lemma 1 and Theorem 1 in [7] are fulfilled. In virtue of these results, Chabrowski [7] established the existence of a weak solution to problem (3) in the case $g \equiv 0$ and $b \equiv \lambda > 0$. We prove in this paper that if we perturbe the problem studied in Chabrowski's paper such that the perturbation does not exceed some level, then equation (3) has at least two distinct solutions. More precisely, if g is small then there is a local minimum near the origin, while the second solution is obtained as a mountain pass. Assumptions (B), (K) and (M) will be used to deduce the existence of the mountain pass solution, while the existence of a simple solution (the local minimum) will follow without these stronger hypotheses. Results of this type have been originally proven in Tarantello [15], but in a different framework. More precisely, Tarantello considered the non-degenerate ($a \equiv 1$) problem (3) in a bounded domain, and for $p = 2^*$ ($N \geq 3$), $b \equiv 0$, $K \equiv 1$ it is showed that (3) has at least two distinct solutions, provided that $g \not\equiv 0$ is sufficiently "small" in a suitable sense.

Our main result is the following.

Theorem 1 *Assume conditions $(A1)$, $(A2)$, (B), (K) and (M) are fulfilled. Then there exists $C > 0$ such that problem (3) has at least two solutions, for any $g \not\equiv 0$ satisfying $\|g\|_{-1} < C$.*

2 Auxiliary results

Weak solutions of (3) correspond to the critical points of the energy functional

$$J(u)=\frac{1}{2}\int_{\mathbf{R}^N}(a(x)|\nabla u|^2+b(x)u^2)\,dx-\frac{1}{p}\int_{\mathbf{R}^N}K(x)|u|^p\,dx-\int_{\mathbf{R}^N}g(x)udx\,,\qquad u\in H^1_{a,b}(\mathbf{R}^N).$$

It is easy to observe that the boundedness of a and b implies that $H^1(\mathbf{R}^N)$ is continuously embedded in $H^1_{a,b}(\mathbf{R}^N)$. Our first result shows that $H^1_{a,b}(\mathbf{R}^N)$ is continuously embedded in $L^p(\mathbf{R}^N)$. Using this fact and (K) we conclude that the functional J is well defined.

Proposition 1 *There exists a positive constant $C_p>0$ such that, for any $u\in H^1_{a,b}(\mathbf{R}^N)$,*

$$\left(\int_{\mathbf{R}^N}|u|^p\,dx\right)^{\frac{1}{p}}\le C_p\left(\int_{\mathbf{R}^N}(a(x)|\nabla u|^2+b(x)u^2)\,dx\right)^{\frac{1}{2}}.$$

Proof. We follow the method used in the proof of Proposition 2.1 in Passaseo [12] (see also Chabrowski [7]). In view of our hypotheses $(A1)$ and $(A2)$, we may assume, by taking R_0 large enough, that

$$\{x;\,a(x)=0\}\subset B(0,R_0-1)\quad\text{and}\quad\inf_{\mathbf{R}^N\setminus B(0,R_0-1)}a(x)>0.\tag{4}$$

Choosing q appearing in (A1), we define $r=\frac{2q}{q+1}$. We see that our hypothesis $q>\frac{Np}{2N+2p-Np}$ implies $p<\frac{Nr}{N-r}$, where $1<r<2\le N$. So, by the Sobolev embedding theorem, $W^{1,r}_0(B(0,R_0))$ is continuously embedded in $L^p(B(0,R_0))$. Using this fact, $(A1)$ and Hölder's inequality we find

$$\begin{aligned}\left(\int_{B(0,R_0)}|u|^p\,dx\right)^{\frac{1}{p}}&\le C_1\left(\int_{B(0,R_0)}|\nabla u|^r\,dx\right)^{\frac{1}{r}}=C_1\left(\int_{B(0,R_0)}\frac{1}{a(x)^{\frac{q}{q+1}}}|\nabla u|^r a(x)^{\frac{q}{q+1}}\,dx\right)^{\frac{1}{r}}\le\\ C_1\left(\int_{B(0,R_0)}\frac{1}{a(x)^q}\,dx\right)^{\frac{1}{2q}}&\left(\int_{B(0,R_0)}a(x)|\nabla u|^2\,dx\right)^{\frac{1}{2}}\le C_2\left(\int_{B(0,R_0)}(a(x)|\nabla u|^2+b(x)u^2)\,dx\right)^{\frac{1}{2}}.\end{aligned}\tag{5}$$

Let $\Psi_{R_0-1}\in C^1(\mathbf{R}^N)$ be such that $\Psi_{R_0-1}=1$ in $\mathbf{R}^N\setminus B(0,R_0)$, $\Psi_{R_0-1}=0$ on $B(0,R_0-1)$ and $0\le\Psi_{R_0-1}\le1$ in $\mathbf{R}^N$. The continuous embedding $H^1(\mathbf{R}^N)\subset L^p(\mathbf{R}^N)$ and relation (4) imply

$$
\begin{gathered}
\left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0)} |u|^p\,dx\right)^{\frac{1}{p}} = \left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0)} |u\Psi_{R_0-1}|^p\,dx\right)^{\frac{1}{p}} \leq \\
\left(\int\limits_{\mathbf{R}^N} |u\Psi_{R_0-1}|^p\,dx\right)^{\frac{1}{p}} \leq C_3\left(\int\limits_{\mathbf{R}^N} (|\nabla(u\Psi_{R_0-1})|^2 + |u\Psi_{R_0-1}|^2)\,dx\right)^{\frac{1}{2}} \leq \\
C_4\left(\int\limits_{\mathbf{R}^N} (|\nabla\Psi_{R_0-1}|^2 u^2 + |\Psi_{R_0-1}|^2|\nabla u|^2 + |\Psi_{R_0-1}|^2 u^2)\,dx\right)^{\frac{1}{2}} \leq \\
C_5\left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0-1)} (|\nabla u|^2 + u^2)\,dx\right)^{\frac{1}{2}} \leq C_6\left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0-1)} (a(x)|\nabla u|^2 + b(x)u^2)\,dx\right)^{\frac{1}{2}},
\end{gathered}
\tag{6}
$$

where C_i with $i = \overline{1,\,6}$ are some positive constants.

From (5), (6) and the elementary inequality

$$
(a+b)^{1/p} \leq C(p)\,(a^{1/p} + b^{1/p}) \qquad \text{for all } a, b > 0
$$

we obtain

$$
\begin{gathered}
\left(\int\limits_{\mathbf{R}^N} |u|^p\,dx\right)^{\frac{1}{p}} \leq C(p)\left[\left(\int\limits_{B(0,R_0)} |u|^p\,dx\right)^{\frac{1}{p}} + \left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0)} |u|^p\,dx\right)^{\frac{1}{p}}\right] \leq \\
C_p\left(\int\limits_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\,dx\right)^{\frac{1}{2}},
\end{gathered}
$$

for some positive constants $C(p)$ and C_p depending only on p. This completes our proof. $\square$

In this paper we denote by "$\rightharpoonup$" the weak convergence and by "$\rightarrow$" the strong convergence, in an arbitrary Banach space X.

Remark 1 Let $\{u_n\}$ be a sequence that converges weakly to some u_0 in $H^1_{a,b}(\mathbf{R}^N)$. Since $\{u_n\}$ is bounded in $H^1_{a,b}(\mathbf{R}^N)$ we see easily that $\{u_n\}$ restricted to $\mathbf{R}^N \setminus B(0, R_0)$ is bounded in $H^1(\mathbf{R}^N \setminus B(0, R_0))$. It also follows from the proof of Proposition 1 that the sequence $\{u_n\}$ restricted to $B(0, R_0)$ is bounded in $W_0^{1,r}(B(0, R_0))$, $p < \frac{Nr}{N-r}$. Therefore, we may assume (up to a subsequence) that

$$
u_n \to u_0 \text{ in } L^p_{\mathrm{loc}}(\mathbf{R}^N) \quad \text{and} \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N. \tag{7}
$$

Remark 2 If we examine carefully the proof of Proposition 1 we see that it holds in order to conclude that $H^1_{a,b}(\mathbf{R}^N)$ is continuously embedded in $L^s(\mathbf{R}^N)$, for every $2 \leq s \leq p$. If $\{u_n\}$ is a bounded sequence in $H^1_{a,b}(\mathbf{R}^N)$, then using the fact that $H^1_{a,b}(\mathbf{R}^N)$ is a reflexive space and Remark 1 we can assume (passing eventually to subsequences) that

$$u_n \rightharpoonup u_0 \text{ in } H^1_{a,b}(\mathbf{R}^N), \quad u_n \to u_0 \text{ in } L^s_{\text{loc}}(\mathbf{R}^N),\ 2 \leq s \leq p \quad \text{and} \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N. \tag{8}$$

We define the functionals $I : H^1_{a,b}(\mathbf{R}^N) \to \mathbf{R}$ and $I_\infty : H^1_{a,b}(\mathbf{R}^N) \to \mathbf{R}$ by

$$I(u) = \frac{1}{2} \int\limits_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\, dx - \frac{1}{p} \int\limits_{\mathbf{R}^N} K(x)|u|^p\, dx,$$
$$I_\infty(u) = \frac{1}{2} \int\limits_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx - \frac{1}{p} \int\limits_{\mathbf{R}^N} K(\infty)|u|^p\, dx.$$

A simple calculation shows that $J,\ I,\ I_\infty \in C^1(H^1_{a,b}(\mathbf{R}^N), \mathbf{R})$ and their derivatives are given by

$$\langle J'(u), v\rangle = \int\limits_{\mathbf{R}^N} (a(x)\nabla u \cdot \nabla v + b(x)uv)\, dx - \int\limits_{\mathbf{R}^N} K(x)|u|^{p-2}uv\, dx - \int\limits_{\mathbf{R}^N} g(x)v\, dx,$$
$$\langle I'(u), v\rangle = \int\limits_{\mathbf{R}^N} (a(x)\nabla u \cdot \nabla v + b(x)uv)\, dx - \int\limits_{\mathbf{R}^N} K(x)|u|^{p-2}uv\, dx,$$
$$\langle I'_\infty(u), v\rangle = \int\limits_{\mathbf{R}^N} (a(x)\nabla u \cdot \nabla v + b(\infty)uv)\, dx - \int\limits_{\mathbf{R}^N} K(\infty)|u|^{p-2}uv\, dx,$$

for all $u,\ v \in H^1_{a,b}(\mathbf{R}^N)$. We have denoted by $\langle\ ,\ \rangle$ the duality pairing between $H^1_{a,b}(\mathbf{R}^N)$ and $H^{-1}_{a,b}(\mathbf{R}^N)$.

Definition 2 *If F is a C^1 functional on some Banach space X and c is a real number, we say that a sequence $\{u_n\}$ in X is a $(PS)_c$ sequence of F if $F(u_n) \to c$ and $F'(u_n) \to 0$ in X^*.*

We now prove that the weak limit (if exists) of any $(PS)_c$ sequence of J is a solution of problem (3).

Lemma 1 *Let $\{u_n\} \subset H^1_{a,b}(\mathbf{R}^N)$ be a $(PS)_c$ sequence of J for some $c \in \mathbf{R}$. Assume that $\{u_n\}$ converges weakly to some u_0 in $H^1_{a,b}(\mathbf{R}^N)$. Then $J'(u_0) = 0$ i.e. u_0 is a weak solution of problem (3).*

Proof. Consider an arbitrary function $\zeta \in C_0^\infty(\mathbf{R}^N)$ and set $\Omega = \operatorname{supp}\zeta$. Obviously $J'(u_n) \to 0$ in $H^{-1}_{a,b}(\mathbf{R}^N)$ implies $\langle J'(u_n), \zeta\rangle \to 0$ as $n \to \infty$, that is

$$\lim_{n\to\infty} \left(\int_\Omega (a(x)\nabla u_n \cdot \nabla\zeta + b(x)u_n\zeta)\, dx - \int_\Omega K(x)|u_n|^{p-2}u_n\zeta\, dx - \int_\Omega g(x)\zeta\, dx \right) = 0. \tag{9}$$

Since $u_n \rightharpoonup u_0$ in $H^1_{a,b}(\mathbf{R}^N)$ it follows that

$$\lim_{n\to\infty} \int_\Omega (a(x)\nabla u_n \cdot \nabla\zeta + b(x)u_n\zeta)\,dx = \int_\Omega (a(x)\nabla u_0 \cdot \nabla\zeta + b(x)u_0\zeta)\,dx. \tag{10}$$

The boundedness of $\{u_n\}$ in $H^1_{a,b}(\mathbf{R}^N)$ and Proposition 1 show that $\{|u_n|^{p-2}u_n\}$ is a bounded sequence in $L^{p/(p-1)}(\mathbf{R}^N)$. Combining this with the convergence $|u_n|^{p-2}u_n \to |u_0|^{p-2}u_0$ a.e. in $\mathbf{R}^N$ (which is a consequence of (7)) we deduce that $|u_0|^{p-2}u_0$ is the weak limit of the sequence $|u_n|^{p-2}u_n$ in $L^{p/(p-1)}(\mathbf{R}^N)$. So

$$\lim_{n\to\infty} \int_\Omega K(x)|u_n|^{p-2}u_n\zeta\,dx = \int_\Omega K(x)|u_0|^{p-2}u_0\zeta\,dx. \tag{11}$$

From (9), (10) and (11) we deduce that

$$\int_\Omega (a(x)\nabla u_0 \cdot \nabla\zeta + b(x)u_0\zeta)\,dx - \int_\Omega K(x)|u_0|^{p-2}u_0\zeta\,dx - \int_\Omega g(x)\zeta\,dx = 0.$$

By density, this equality holds for any $\zeta \in H^1_{a,b}(\mathbf{R}^N)$ which means that $J'(u_0) = 0$. The proof of our lemma is complete. □

Brezis and Lieb established in [5, Theorem 1] a subtle refinement of Fatou's Lemma. Our next result is a weighted variant of the Brezis-Lieb Lemma.

Lemma 2 *Let $\{u_n\}$ be a sequence which is weakly convergent to u_0 in $H^1_{a,b}(\mathbf{R}^N)$. Then*

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} K(x)\left(|u_n|^p - |u_n - u_0|^p\right)dx = \int_{\mathbf{R}^N} K(x)|u_0|^p\,dx.$$

Proof. From Proposition 1 and the boundedness of $\{u_n\}$ in $H^1_{a,b}(\mathbf{R}^N)$ we obtain that $\{u_n\}$ is a bounded sequence in $L^p(\mathbf{R}^N)$. For a given $\varepsilon > 0$ we choose $R_\varepsilon > 0$ such that

$$\int_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx < \varepsilon. \tag{12}$$

We have

$$\begin{aligned}
&\Big|\int_{\mathbf{R}^N} K(x)(|u_n|^p - |u_0|^p - |u_n-u_0|^p)\,dx\Big| = \Big|\int_{|x|\le R_\varepsilon} K(x)(|u_n|^p - |u_0|^p)\,dx - \\
&\int_{|x|\le R_\varepsilon} K(x)|u_n-u_0|^p\,dx - \int_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx + \int_{|x|>R_\varepsilon} K(x)(|u_n|^p - |u_n-u_0|^p)\,dx\Big| \le \\
&\Big|\int_{|x|\le R_\varepsilon} K(x)(|u_n|^p - |u_0|^p)\,dx\Big| + \int_{|x|\le R_\varepsilon} K(x)|u_n-u_0|^p\,dx + \int_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx + \\
&\int_{|x|>R_\varepsilon} p\,K(x)|\theta u_0 + (u_n-u_0)|^{p-1}|u_0|\,dx, \quad \text{where} \quad 0 \le \theta(x) \le 1.
\end{aligned} \tag{13}$$

From (12) and Hölder's inequality we find

$$\int\limits_{|x|>R_\varepsilon} K(x)|\theta u_0 + (u_n - u_0)|^{p-1}|u_0|\,dx \le c \int\limits_{|x|>R_\varepsilon} K(x)(|u_0|^p + |u_n - u_0|^{p-1}|u_0|)\,dx \le$$

$$c\left[\int\limits_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx + \left(\int\limits_{|x|>R_\varepsilon} K(x)|u_n-u_0|^p\,dx\right)^{\frac{p-1}{p}}\left(\int\limits_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx\right)^{\frac{1}{p}}\right] < \tilde{c}(\varepsilon + \varepsilon^{\frac{1}{p}}) \tag{14}$$

for some constants $c, \tilde{c} > 0$ independent of n and ε.
Now, by (7),

$$\lim_{n\to\infty} \int\limits_{|x|\le R_\varepsilon} K(x)(|u_n|^p - |u_0|^p)\,dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int\limits_{|x|\le R_\varepsilon} K(x)|u_n - u_0|^p\,dx = 0. \tag{15}$$

From (12), (13), (14) and (15) it follows that

$$\limsup_{n\to\infty} \Big|\int\limits_{\mathbf{R}^N} K(x)(|u_n|^p - |u_0|^p - |u_n-u_0|^p)\,dx\Big| \le (p\,\tilde{c} + 1)(\varepsilon + \varepsilon^{\frac{1}{p}})\,.$$

Since $\varepsilon > 0$ is arbitrary we deduce that

$$\lim_{n\to\infty}\left(\int\limits_{\mathbf{R}^N} K(x)|u_n|^p\,dx - \int\limits_{\mathbf{R}^N} K(x)|u_0|^p\,dx - \int\limits_{\mathbf{R}^N} K(x)|u_n-u_0|^p\,dx\right) = 0,$$

which concludes our proof. □

Lemma 3 *Let $\{v_n\} \subset H^1_{a,b}(\mathbf{R}^N)$ be a sequence converging weakly to 0 in $H^1_{a,b}(\mathbf{R}^N)$. Then*

$$\lim_{n\to\infty}[I(v_n) - I_\infty(v_n)] = 0; \tag{16}$$

$$\lim_{n\to\infty}[\langle I'(v_n), v_n\rangle - \langle I'_\infty(v_n), v_n\rangle] = 0. \tag{17}$$

Proof. A simple computation yields

$$I(v_n) = I_\infty(v_n) - \frac{1}{2}\int\limits_{\mathbf{R}^N}(b(\infty) - b(x))v_n^2\,dx - \frac{1}{p}\int\limits_{\mathbf{R}^N}(K(x) - K(\infty))|v_n|^p\,dx.$$

$$\langle I'(v_n), v_n\rangle = \langle I'_\infty(v_n), v_n\rangle - \int\limits_{\mathbf{R}^N}(b(\infty) - b(x))v_n^2\,dx - \int\limits_{\mathbf{R}^N}(K(x) - K(\infty))|v_n|^p\,dx.$$

Let $\varepsilon > 0$ be a positive number. The assumption (K) implies that there exists $R_\varepsilon > 0$ such that

$$|K(x) - K(\infty)| = K(x) - K(\infty) < \varepsilon \quad \text{for a.e. } x \in \mathbf{R}^N \text{ with } |x| > R_\varepsilon.$$

Using this fact we obtain

$$\int_{\mathbf{R}^N} (K(x) - K(\infty))|v_n|^p \, dx = \int_{|x| \le R_\varepsilon} (K(x) - K(\infty))|v_n|^p \, dx + \int_{|x| > R_\varepsilon} (K(x) - K(\infty))|v_n|^p \, dx \le$$
$$(\|K\|_\infty - K(\infty)) \int_{|x| \le R_\varepsilon} |v_n|^p \, dx + \varepsilon \left(\int_{|x| > R_\varepsilon} |v_n|^p \, dx \right).$$

Since $v_n \rightharpoonup 0$ in $H^1_{a,b}(\mathbf{R}^N)$, it follows by Proposition 1 that $\{v_n\}$ is bounded in $L^p(\mathbf{R}^N)$. On the other hand, in virtue of (7), we have that $v_n \to 0$ in $L^p_{\text{loc}}(\mathbf{R}^N)$. Then letting $n \to \infty$ we see that

$$\limsup_{n \to \infty} \int_{\mathbf{R}^N} (K(x) - K(\infty))|v_n|^p \, dx \le C\,\varepsilon$$

for some constant $C > 0$ independent of n and ε. It follows that

$$\lim_{n \to \infty} \int_{\mathbf{R}^N} (K(x) - K(\infty))|v_n|^p \, dx = 0.$$

To prove (16) and (17) we need only to show that

$$\lim_{n \to \infty} \int_{\mathbf{R}^N} (b(\infty) - b(x)) v_n^2 \, dx = 0. \tag{18}$$

To this end, notice that for any $R > 0$ we have

$$\begin{aligned} \int_{\mathbf{R}^N} (b(\infty) - b(x)) v_n^2 \, dx = \int_{|x| \le R} (b(\infty) - b(x)) v_n^2 \, dx + \int_{|x| > R} (b(\infty) - b(x)) v_n^2 \, dx \le \\ (b(\infty) - b_1) \int_{|x| \le R} v_n^2 \, dx + \int_{|x| > R} (b(\infty) - b(x)) v_n^2 \, dx. \end{aligned} \tag{19}$$

From (B) we have that for any $\varepsilon > 0$ we can choose $R_\varepsilon > 0$ such that

$$|b(\infty) - b(x)| = b(\infty) - b(x) < \varepsilon \quad \text{for a.e. } x \in \mathbf{R}^N \text{ with } |x| > R_\varepsilon. \tag{20}$$

But, from Remark 2, we know that $H^1_{a,b}(\mathbf{R}^N)$ is continuously embedded in $L^2(\mathbf{R}^N)$ and, by (8), $v_n \to 0$ in $L^2_{\text{loc}}(\mathbf{R}^N)$. Therefore, using (19) and (20) we deduce the existence of a positive number M, independent of n and ε, such that

$$\limsup_{n \to \infty} \int_{\mathbf{R}^N} (b(\infty) - b(x)) v_n^2 \, dx \le M\,\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that (18) is true. □

Lemma 4 *For any $0 < \varepsilon < 1$ there exist $R = R(\varepsilon) > 0$ and $C = C(\varepsilon) > 0$ such that for all g with $\|g\|_{-1} < C$, there exists a $(PS)_{c_0}$ sequence of $J(u)$ with $c_0 = c_0(R) = \inf\limits_{u \in \overline{B}_R} J(u)$, where $\overline{B}_R = \{u \in H^1_{a,b}(\mathbf{R}^N);\ \|u\|_{a,b} \le R\}$. Moreover, $c_0(R)$ is achieved by some $u_0 \in H^1_{a,b}(\mathbf{R}^N)$ with $J'(u_0) = 0$.*

Proof. Fix $0 < \varepsilon < 1$. Then for any $u \in H^1_{a,b}(\mathbf{R}^N)$, by (K) and Young's inequality we have

$$\begin{aligned}
&J(u) = \frac{1}{2}\|u\|^2_{a,b} - \frac{1}{p}\int\limits_{\mathbf{R}^N} K(x)|u|^p\,dx - \int\limits_{\mathbf{R}^N} g(x)\,u\,dx \ge \\
&\frac{1}{2}\|u\|^2_{a,b} - \frac{\|K\|_\infty}{p}\|u\|^p_{L^p(\mathbf{R}^N)} - \|u\|_{a,b}\|g\|_{-1} \ge \\
&\frac{1}{2}\|u\|^2_{a,b} - \frac{\|K\|_\infty}{p}C_0^p\|u\|^p_{a,b} - \left(\frac{\varepsilon^2}{2}\|u\|^2_{a,b} + \frac{1}{2\varepsilon^2}\|g\|^2_{-1}\right) = \\
&\left(\frac{1}{2} - \frac{\varepsilon^2}{2}\right)\|u\|^2_{a,b} - \frac{\|K\|_\infty}{p}C_0^p\|u\|^p_{a,b} - \frac{1}{2\varepsilon^2}\|g\|^2_{-1}\,,
\end{aligned}$$

where $C_0 > 0$ is a positive constant given by Proposition 1. The above estimate shows the existence of $R = R(\varepsilon) > 0$, $C = C(\varepsilon) > 0$ and $\delta = \delta(R) > 0$ such that $J(u)\ |_{\partial B_R} \ge \delta > 0$ for all g with $\|g\|_{-1} \le C$. For example, we can take

$$R(\varepsilon) = \left(\frac{1-\varepsilon^2}{\|K\|_\infty C_0^p}\right)^{\frac{1}{p-2}},\ C(\varepsilon) = \sqrt{M}\varepsilon,\ \delta(R) = \frac{M}{2}, \quad \text{where} \quad M = M(R) = \left(\frac{1}{2} - \frac{1}{p}\right)\|K\|_\infty C_0^p R^p.$$

Define $c_0 = c_0(R) = \inf\limits_{u \in \overline{B}_R} J(u)$. So, $c_0 \le J(0) = 0$. The set $\overline{B}_R$ becomes a complete metric space with respect to the distance

$$\operatorname{dist}(u,v) = \|u - v\|_{a,b} \quad \text{for any } u, v \in \overline{B}_R.$$

On the other hand, J is lower semi-continuous and bounded from below on $\overline{B}_R$. So, by Ekeland's Variational Principle [8, Theorem 1.1], for any positive integer n there exists u_n with

$$c_0 \le J(u_n) \le c_0 + \frac{1}{n}, \tag{21}$$

$$J(w) \ge J(u_n) - \frac{1}{n}\|u_n - w\|_{a,b} \quad \text{for all } w \in \overline{B}_R. \tag{22}$$

We claim that $\|u_n\|_{a,b} < R$ for n large enough. Indeed, if $\|u_n\|_{a,b} = R$ for infinitely many n, we may assume, without loss of generality, that $\|u_n\|_{a,b} = R$ for all $n \ge 1$. It follows that $J(u_n) \ge \delta > 0$. Combining this with (21) and letting $n \to \infty$, we have $0 \ge c_0 \ge \delta > 0$ which is a contradiction.

We now prove that $J'(u_n) \to 0$ in $H^{-1}_{a,b}(\mathbf{R}^N)$. Indeed, for any $u \in H^1_{a,b}(\mathbf{R}^N)$ with $\|u\|_{a,b} = 1$, let $w_n = u_n + tu$. For a fixed n, we have $\|w_n\|_{a,b} \leq \|u_n\|_{a,b} + t < R$, where $t > 0$ is small enough. Using (22) we obtain

$$J(u_n + tu) \geq J(u_n) - \frac{t}{n}\|u\|_{a,b}$$

that is

$$\frac{J(u_n + tu) - J(u_n)}{t} \geq -\frac{1}{n}\|u\|_{a,b} = -\frac{1}{n}.$$

Letting $t \searrow 0$, we deduce that $\langle J'(u_n), u\rangle \geq -\dfrac{1}{n}$ and a similar argument for $t \nearrow 0$ produces $|\langle J'(u_n), u\rangle| \leq \dfrac{1}{n}$ for any $u \in H^1_{a,b}(\mathbf{R}^N)$ with $\|u\|_{a,b} = 1$. So,

$$\|J'(u_n)\|_{-1} = \sup_{\substack{u \in H^1_{a,b}(\mathbf{R}^N) \\ \|u\|_{a,b}=1}} |\langle J'(u_n), u\rangle| \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

We have obtained the existence of a $(PS)_{c_0}$ sequence, i.e. a sequence $\{u_n\} \subset H^1_{a,b}(\mathbf{R}^N)$ with

$$J(u_n) \to c_0 \quad \text{and} \quad J'(u_n) \to 0 \text{ in } H^{-1}_{a,b}(\mathbf{R}^N). \tag{23}$$

But $\|u_n\|_{a,b} \leq R$, for the fixed R, shows that $\{u_n\}$ converges weakly (up to a subsequence) in $H^1_{a,b}(\mathbf{R}^N)$. Therefore (7), (23) and Lemma 1 imply that, for some $u_0 \in H^1_{a,b}(\mathbf{R}^N)$

$$u_n \rightharpoonup u_0 \text{ in } H^1_{a,b}(\mathbf{R}^N), \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N \tag{24}$$

$$J'(u_0) = 0. \tag{25}$$

We prove that $J(u_0) = c_0$. By (23) and (24) we have

$$o(1) = \langle J'(u_n), u_n\rangle = \int_{\mathbf{R}^N} (a(x)|\nabla u_n|^2 + b(x)u_n^2)\,dx - \int_{\mathbf{R}^N} K(x)|u_n|^p\,dx - \int_{\mathbf{R}^N} g(x)u_n\,dx.$$

Therefore

$$J(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbf{R}^N} K(x)|u_n|^p\,dx - \frac{1}{2}\int_{\mathbf{R}^N} g(x)u_n\,dx + o(1).$$

By (23), (24), (25) and Fatou's lemma we have

$$c_0 = \liminf_{n\to\infty} J(u_n) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbf{R}^N} K(x)|u_0|^p\,dx - \frac{1}{2}\int_{\mathbf{R}^N} g(x)u_0\,dx = J(u_0).$$

Since $u_0 \in \overline{B}_R$, it follows that $J(u_0) = c_0$. $\square$

3 Proof of Theorem 1

Set

$$S = \{u \in H^1_{a,b}(\mathbf{R}^N) \setminus \{0\};\ \langle I'_\infty(u), u\rangle = 0\}.$$

We first justify that $S \not\equiv \emptyset$. Indeed, fix $u_0 \in H^1_{a,b}(\mathbf{R}^N) \setminus \{0\}$ and set, for any $\lambda > 0$,

$$\Psi(\lambda) = \langle I'_\infty(\lambda u_0), \lambda u_0\rangle\,.$$

It follows that

$$\Psi(\lambda) = \lambda^2 \left(\int_{\mathbf{R}^N} \left(a(x)|\nabla u_0|^2 + b(\infty)u_0^2\right)\, dx - \lambda^{p-2}\int_{\mathbf{R}^N} K(\infty)|u_0|^p\, dx\right)\,.$$

Our hypotheses imply that $\Psi(\lambda) < 0$ for λ large enough and $\Psi(\lambda) > 0$ for λ sufficiently close to zero. It follows that there exists $\lambda_0 \in (0,\infty)$ such that $\Psi(\lambda_0) = 0$. This means that $\lambda_0 u_0 \in S$.

Proposition 2 *Let* $J_\infty = \inf\{I_\infty(u);\ u \in S\}$. *Then there exists* $\bar{u} \in H^1_{a,b}(\mathbf{R}^N)$ *such that*

$$J_\infty = I_\infty(\bar{u}) = \sup_{t\geq 0} I_\infty(t\bar{u}). \tag{26}$$

Proof. We consider the constrained minimization problem

$$m = \inf\left\{\int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx;\ u \in H^1_{a,b}(\mathbf{R}^N),\ \int_{\mathbf{R}^N} K(\infty)|u|^p\, dx = 1\right\}. \tag{27}$$

For every $\varphi \in H^1_{a,b}(\mathbf{R}^N) \setminus \{0\}$ let

$$f(t) = I_\infty(t\varphi) = \frac{t^2}{2}\int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx - \frac{t^p}{p}\int_{\mathbf{R}^N} K(\infty)|\varphi|^p\, dx.$$

We have

$$f'(t) = t\int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx - t^{p-1}\int_{\mathbf{R}^N} K(\infty)|\varphi|^p\, dx,$$

which vanishes for

$$\bar{t}(\varphi) = \left\{\frac{\displaystyle\int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx}{\displaystyle\int_{\mathbf{R}^N} K(\infty)|\varphi|^p\, dx}\right\}^{\frac{1}{p-2}}.$$

Hence

$$f(\bar{t}(\varphi)) = I_\infty(\bar{t}(\varphi)\varphi) = \sup_{t\geq 0} I_\infty(t\varphi) = \left(\frac{1}{2}-\frac{1}{p}\right)\left\{\frac{\int_{\mathbf{R}^N}(a(x)|\nabla u|^2+b(\infty)u^2)\,dx}{\left(\int_{\mathbf{R}^N}K(\infty)|\varphi|^p\,dx\right)^{\frac{2}{p}}}\right\}^{\frac{p}{p-2}}.$$

It follows that

$$\inf_{\varphi\in H^1_{a,b}(\mathbf{R}^N)\setminus\{0\}}\ \sup_{t\geq 0} I_\infty(t\varphi) = \left(\frac{1}{2}-\frac{1}{p}\right)m^{\frac{p}{p-2}}.$$

We easily observe that for every $u\in S$ we have $\bar{t}(u)=1$ which implies $I_\infty(u)=\sup_{t\geq 0} I_\infty(tu)$. Let $\{u_n\}\subset H^1_{a,b}(\mathbf{R}^N)$ be a minimizing sequence for problem (27), i.e.,

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}(a(x)|\nabla u_n|^2+b(\infty)u_n^2)\,dx = m \quad\text{and}\quad \int_{\mathbf{R}^N}K(\infty)|u_n|^p\,dx = 1.$$

Then $v_n = m^{\frac{1}{p-2}}u_n$ satisfies

$$\begin{aligned} &(i) \quad I_\infty(v_n)\to\left(\tfrac{1}{2}-\tfrac{1}{p}\right)m^{\frac{p}{p-2}} &&\text{as } n\to\infty\\ &(ii) \quad I'_\infty(v_n)\to 0 \quad\text{in } H^{-1}_{a,b}(\mathbf{R}^N) &&\text{as } n\to\infty\,. \end{aligned}$$

Now, using (B) we get that the minimizing sequence $\{u_n\}$ is bounded in $H^1_{a,b}(\mathbf{R}^N)$ and, by Remark 2, we find $u\in H^1_{a,b}(\mathbf{R}^N)$ such that (up to a subsequence) $u_n\rightharpoonup u$ in $H^1_{a,b}(\mathbf{R}^N)$ and $u_n\to u$ in $L^p_{\rm loc}(\mathbf{R}^N)$. Our hypotheses $(A1)$ and $(A2)$ allow us to apply Lemma 1 and Theorem 1 in [7] in order to find that $u\not\equiv 0$ and u is a solution of problem (27). Letting $\bar{u}=m^{\frac{1}{p-2}}u$, we see that $\bar{u}\in S$ and $I_\infty(\bar{u}) = \left(\frac{1}{2}-\frac{1}{p}\right)m^{\frac{p}{p-2}}$. We obtain

$$J_\infty = \inf_{u\in S} I_\infty(u) = \inf_{u\in S}\sup_{t\geq 0} I_\infty(tu) \geq \inf_{u\in H^1_{a,b}(\mathbf{R}^N)\setminus\{0\}}\ \sup_{t\geq 0} I_\infty(tu) = \left(\frac{1}{2}-\frac{1}{p}\right)m^{\frac{p}{p-2}} = I_\infty(\bar{u})$$

which concludes our proof. □

Proposition 3 *Assume $\{u_n\}$ is a $(PS)_c$ sequence of J that converges weakly to u_0 in $H^1_{a,b}(\mathbf{R}^N)$. Then either $\{u_n\}$ converges strongly in $H^1_{a,b}(\mathbf{R}^N)$, or $c\geq J(u_0)+J_\infty$.*

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence and $u_n\rightharpoonup u_0$ in $H^1_{a,b}(\mathbf{R}^N)$ we have

$$J(u_n) = c+o(1) \quad\text{and}\quad \langle J'(u_n),u_n\rangle = o(1)\,. \tag{28}$$

Set $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in $H^1_{a,b}(\mathbf{R}^N)$ which implies

$$\int_{\mathbf{R}^N} (a(x)\nabla v_n \nabla u_0 + b(x) v_n u_0)\, dx \to 0 \quad \text{as } n \to \infty\,,$$
$$\int_{\mathbf{R}^N} g(x) v_n\, dx \to 0 \quad \text{as } n \to \infty\,.$$

We rewrite the above relations as

$$\begin{aligned} \|u_n\|^2_{a,b} &= \|u_0\|^2_{a,b} + \|v_n\|^2_{a,b} + o(1)\,, \\ J(v_n) &= I(v_n) + o(1)\,. \end{aligned} \tag{29}$$

From (28), (29), Lemma 1 and Lemma 2 it follows that

$$\begin{aligned} o(1) + c = J(u_n) &= J(u_0) + J(v_n) + o(1) = J(u_0) + I(v_n) + o(1)\,, \\ o(1) = \langle J'(u_n), u_n\rangle &= \langle J'(u_0), u_0\rangle + \langle J'(v_n), v_n\rangle + o(1) = \langle I'(v_n), v_n\rangle + o(1)\,. \end{aligned} \tag{30}$$

If $v_n \to 0$ in $H^1_{a,b}(\mathbf{R}^N)$, then $u_n \to u_0$ in $H^1_{a,b}(\mathbf{R}^N)$ and $J(u_0) = \lim_{n\to\infty} J(u_n) = c$.

If $v_n \not\to 0$ in $H^1_{a,b}(\mathbf{R}^N)$, then combining this with the fact that $v_n \rightharpoonup 0$ in $H^1_{a,b}(\mathbf{R}^N)$ we may assume that $\|v_n\|_{a,b} \to l > 0$. Then (30) and Lemma 3 imply

$$c = J(u_0) + I_\infty(v_n) + o(1) \tag{31}$$

$$\mu_n = \langle I'_\infty(v_n), v_n\rangle = \int_{\mathbf{R}^N} (a(x)|\nabla v_n|^2 + b(\infty) v_n^2)\, dx - \int_{\mathbf{R}^N} K(\infty)|v_n|^p\, dx = \alpha_n - \beta_n, \tag{32}$$

where $\lim_{n\to\infty} \mu_n = 0$, $\alpha_n = \int_{\mathbf{R}^N} (a(x)|\nabla v_n|^2 + b(\infty) v_n^2)\, dx \geq \|v_n\|^2_{a,b}$ and $\beta_n = \int_{\mathbf{R}^N} K(\infty)|v_n|^p\, dx \geq 0$. In virtue of (31), it remains to show that $I_\infty(v_n) \geq J_\infty + o(1)$. For $t > 0$, we have

$$\langle I'_\infty(tv_n), tv_n\rangle = t^2 \int_{\mathbf{R}^N} (a(x)|\nabla v_n|^2 + b(\infty) v_n^2)\, dx - t^p \int_{\mathbf{R}^N} K(\infty)|v_n|^p\, dx.$$

If we prove the existence of a sequence $\{t_n\}$ with $t_n > 0$, $t_n \to 1$ and $\langle I'_\infty(t_n v_n), t_n v_n\rangle = 0$, then

$$I_\infty(v_n) = I_\infty(t_n v_n) + \frac{1-t_n^2}{2}\alpha_n - \frac{1-t_n^p}{p} K(\infty)\|v_n\|^p_{L^p(\mathbf{R}^N)} = I_\infty(t_n v_n) + o(1) \geq J_\infty + o(1)$$

and the conclusion follows. To do this, let $t = 1 + \delta$ with $|\delta|$ small enough and using (32) we obtain

$$\begin{aligned} &\langle I'_\infty(tv_n), tv_n\rangle = (1+\delta)^2\alpha_n - (1+\delta)^p\beta_n = (1+\delta)^2\alpha_n - (1+\delta)^p(\alpha_n - \mu_n) = \\ &\alpha_n(2\delta - p\delta + o(\delta)) + (1+\delta)^p\mu_n = \alpha_n(2-p)\delta + \alpha_n o(\delta) + (1+\delta)^p\mu_n. \end{aligned}$$

Since $\alpha_n \to \bar{l} \geq l^2 > 0$, $\lim_{n\to\infty} \mu_n = 0$ and $p > 2$ then, for n large enough, we can define $\delta_n^+ = \dfrac{2\,|\mu_n|}{\alpha_n(p-2)}$ and $\delta_n^- = \dfrac{-2\,|\mu_n|}{\alpha_n(p-2)}$ which satisfy the following properties

$$\begin{aligned} &\delta_n^+ \searrow 0 \quad \text{and} \quad \langle I'_\infty((1+\delta_n^+)v_n), (1+\delta_n^+)v_n\rangle < 0\,, \\ &\delta_n^- \nearrow 0 \quad \text{and} \quad \langle I'_\infty((1+\delta_n^-)v_n), (1+\delta_n^-)v_n\rangle > 0\,. \end{aligned} \tag{33}$$

From (33) we deduce the existence of $t_n \in (1+\delta_n^-, 1+\delta_n^+)$ such that

$$t_n \to 1 \quad \text{and} \quad \langle I'_\infty(t_n v_n), t_n v_n\rangle = 0.$$

This concludes our proof. □

Let $\bar{u} \in H^1_{a,b}(\mathbf{R}^N)$ be such that (26) holds. We can find $\bar{t} > 0$ such that

$$\begin{aligned} &I(t\bar{u}) < 0 \quad \text{if } t \geq \bar{t} \\ &J(t\bar{u}) < 0 \quad \text{if } t \geq \bar{t} \text{ and } \|g\|_{-1} \leq 1\,. \end{aligned}$$

We put

$$\mathcal{P} = \{\gamma \in C([0,1], H^1_{a,b}(\mathbf{R}^N));\ \gamma(0) = 0, \gamma(1) = \bar{t}\bar{u}\} \tag{34}$$

$$c_g = \inf_{\gamma\in\mathcal{P}} \sup_{u\in\gamma} J(u). \tag{35}$$

Proposition 4 *There exist $R_0 > 0$, $C = C(R_0) > 0$ and $\delta_{R_0} > 0$ such that for all g with $\|g\|_{-1} < C$ we have $J\,|_{\partial B_{R_0}} \geq \delta_{R_0}$ and $c_g < c_0 + J_\infty$, where c_g is given by (35) and $c_0 = \inf_{u\in\overline{B}_{R_0}} J(u)$.*

Proof. By our hypothesis (M) and the definition of I we can assume that $I(t\bar{u}) < I_\infty(t\bar{u})$ for all $t > 0$. A simple computation implies that there exists $t_0 \in (0, \bar{t})$ such that

$$\sup_{t\geq 0} I(t\bar{u}) = I(t_0\bar{u}) < I_\infty(t_0\bar{u}) \leq \sup_{t\geq 0} I_\infty(t\bar{u}) = J_\infty.$$

Then there exists an $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\geq 0} I(t\bar{u}) < J_\infty - \varepsilon_0. \tag{36}$$

For this ε_0, we get the existence of $R_0 = R_0(\varepsilon_0)$ and $C_1 = C_1(\varepsilon_0) = C_1(R_0)$ such that for all g with $\|g\|_{-1} < C_1$ the conclusion of Lemma 4 holds. Moreover, in virtue of its proof, there exists $\delta_{R_0} > 0$ such that $J\,|_{\partial B_{R_0}} \geq \delta_{R_0}$, provided that $\|g\|_{-1} < C_1$. Taking $C_2 = \min\{C_1, \varepsilon_0\sqrt{\varepsilon_0}\}$ we find

$$c_0 = \inf_{u\in\overline{B}_{R_0}} J(u) \geq -\frac{1}{2\varepsilon_0^2}\|g\|_{-1}^2 > -\frac{\varepsilon_0}{2} \quad \text{for all } g \text{ with } \|g\|_{-1} < C_2. \tag{37}$$

If $\|g\|_{-1} < \dfrac{\varepsilon_0}{2\bar{t}\,\|\bar{u}\|_{a,b}}$, then for $u \in \gamma_0 = \{t\bar{t}\,\bar{u};\ 0 \le t \le 1\}$ we have

$$|J(u) - I(u)| = |\int\limits_{\mathbf{R}^N} g(x)u\,dx| \le \bar{t}\,|\int\limits_{\mathbf{R}^N} g(x)\bar{u}\,dx| \le \bar{t}\,\|\bar{u}\|_{a,b}\|g\|_{-1} < \frac{\varepsilon_0}{2}.$$

So, if $\|g\|_{-1} < C = \min\left\{C_2, \dfrac{\varepsilon_0}{2\bar{t}\,\|\bar{u}\|_{a,b}}\right\}$ then for all g with $\|g\|_{-1} < C$ we obtain

$$J(u) < I(u) + \frac{\varepsilon_0}{2} \quad \text{for } u \in \gamma_0$$

and from (34), (36), (37) it follows that

$$c_g = \inf_{\gamma\in\mathcal{P}}\sup_{u\in\gamma} J(u) \le \sup_{u\in\gamma_0} J(u) \le \sup_{u\in\gamma_0} I(u) + \frac{\varepsilon_0}{2} \le \sup_{t\ge 0} I(t\bar{u}) + \frac{\varepsilon_0}{2} < J_\infty - \frac{\varepsilon_0}{2} < J_\infty + c_0.$$

□

Proof of Theorem 1 concluded. Consider $R_0 > 0$, $C = C(R_0) > 0$ and $\delta_{R_0} > 0$ given by Proposition 4 and, in view of its proof, we have that for all g with $\|g\|_{-1} < C$ the conclusion of Lemma 4 is also true. Therefore, we obtain the existence of a solution $u_0 \in H^1_{a,b}(\mathbf{R}^N)$ of (3) such that $J(u_0) = c_0$.

On the other hand, it follows from the Mountain Pass Theorem without the Palais-Smale condition [6, Theorem 2.2] that there is a $(PS)_{c_g}$ sequence $\{u_n\}$ of $J(u)$, that is

$$J(u_n) = c_g + o(1) \quad \text{and} \quad J'(u_n) \to 0 \text{ in } H^{-1}_{a,b}(\mathbf{R}^N)\,.$$

This implies

$$c_g + o(1) + \frac{1}{p}\|J'(u_n)\|_{-1}\|u_n\|_{a,b} \ge J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle \ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2_{a,b} - \left(1 - \frac{1}{p}\right)\|g\|_{-1}\|u_n\|_{a,b}.$$

Hence $\{u_n\}$ is a bounded sequence in $H^1_{a,b}(\mathbf{R}^N)$ and, passing to a subsequence, we may assume that $u_n \rightharpoonup u_1$ in $H^1_{a,b}(\mathbf{R}^N)$ for some $u_1 \in H^1_{a,b}(\mathbf{R}^N)$. So, by Lemma 1, u_1 is a weak solution of (3).

We prove in what follows that $J(u_0) \ne J(u_1)$. Indeed, by Proposition 3, either $u_n \to u_1$ in $H^1_{a,b}(\mathbf{R}^N)$ which gives

$$J(u_1) = \lim_{n\to\infty} J(u_n) = c_g > 0 \ge c_0 = J(u_0)$$

and the conclusion follows, or

$$c_g = \lim_{n\to\infty} J(u_n) \ge J(u_1) + J_\infty.$$

If we suppose that $J(u_1) = J(u_0) = c_0$, then $c_g \ge c_0 + J_\infty$ which contradicts Proposition 4. This concludes our proof. □

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Critical singular problems on infinite cones

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Abstract

We prove existence results for non autonomous perturbations of singular critical elliptic boundary value problems. The non singular case was treated by Tarantello [11] for bounded domains; here the singular weight allows for unbounded domains as cones and give rise to a different non compactness picture (as was first remarked by Caldiroli and Musina [5]).
Keywords : Singular weights, critical exponent, unbounded domains, Caffarelli-Kohn-Nirenberg inequalities.

1 Introduction

Let Ω be an open set in $\mathbb{R}^N$, $N \geq 2$ and let $\alpha \in (0,2)$. For any $\zeta \in C_c^\infty(\Omega)$, define

$$||\zeta||_\alpha = \left(\int_\Omega |x|^\alpha |\nabla \zeta|^2 dx\right)^{1/2}.$$

Let $H_0^1(\Omega; |x|^\alpha)$ be the closure of $C_c^\infty(\Omega)$ with respect to the $||\cdot||_\alpha$-norm. It turns out that $H_0^1(\Omega; |x|^\alpha)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_\alpha = \int_\Omega |x|^\alpha \nabla u \cdot \nabla v dx, \qquad \forall u, v \in H_0^1(\Omega; |x|^\alpha).$$

If $\Omega = \mathbb{R}^N$ we set $H^1(\mathbb{R}^N; |x|^\alpha) = H_0^1(\mathbb{R}^N; |x|^\alpha)$. We remark that if Ω_1 and Ω_2 are arbitrary open sets in $\mathbb{R}^N$ such that $\Omega_1 \subset \Omega_2$ then $H_0^1(\Omega_1; |x|^\alpha) \hookrightarrow H_0^1(\Omega_2; |x|^\alpha)$, with continuous embedding. We also point out that since we allow the cases $0 \in \overline{\Omega}$ or Ω unbounded then there is no inclusion relationship between $H_0^1(\Omega; |x|^\alpha)$ and the standard Sobolev space $H_0^1(\Omega)$. However, the Caffarelli-Kohn-Nirenberg inequality asserts that $H_0^1(\Omega; |x|^\alpha)$ is continuously embedded in $L^{2_\alpha^*}(\Omega)$, where $2_\alpha^* = 2N/(N-2+\alpha)$. More precisely, there exists $C_\alpha > 0$ such that

$$\left(\int_\Omega |u|^{2_\alpha^*} dx\right)^{1/2_\alpha^*} \leq C_\alpha \left(\int_\Omega |x|^\alpha |\nabla u|^2 dx\right)^{1/2},$$

for any $u \in H_0^1(\Omega; |x|^\alpha)$.

Consider the problem

$$\begin{cases} -\mathrm{div}\,(|x|^\alpha \nabla u) = |u|^{2_\alpha^*-2}u & \text{in } \Omega\,, \\ u \geq 0, \;\; u \not\equiv 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

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We observe that degeneracy occurs in (1) if $0 \in \overline{\Omega}$ or if Ω is unbounded. We also point out that if 2_α^* in problem (1) is replaced by a subcritical exponent $p \in [2, 2_\alpha^*)$ then the corresponding equation is characterized by local compactness, and existence results are carried out in an easier way.

Consider the quotient

$$S_\alpha(u;\Omega) = \frac{\int_\Omega |x|^\alpha |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2_\alpha^*} dx\right)^{2/2_\alpha^*}},$$

and denote

$$S_\alpha(\Omega) = \inf_{u \in H_0^1(\Omega;|x|^\alpha)\setminus\{0\}} S_\alpha(u;\Omega). \tag{2}$$

It is obvious that if $u \in H_0^1(\Omega; |x|^\alpha)$ satisfies

$$\int_\Omega |x|^\alpha |\nabla u|^2 dx = S_\alpha(\Omega) \qquad \text{and} \qquad \int_\Omega |u|^{2_\alpha^*} dx = 1,$$

then the function $U(x) = [S_\alpha(\Omega)]^{1/(2_\alpha^*-2)}\, u(x)$ is a solution of (1).

Caldiroli and Musina [5] studied the critical case and they showed that some concentration phenomena may occur in (1), due to the action of the non compact group of dilations in $\mathbb{R}^N$. They proved in [5] that if $\alpha \in (0,2)$ then, in certain cases, $S_\alpha(\Omega)$ is attained in $H_0^1(\Omega;|x|^\alpha)$ by a positive function, so problem (1) has a solution. We point out (see Struwe [10, Theorem III.1.2]) that $S_\alpha(\Omega)$ is never attained in $H_0^1(\Omega)$ in the limiting case $\alpha = 0$ and if $\Omega \neq \mathbb{R}^N$.

Let $H^{-1}(\Omega;|x|^\alpha)$ be the dual space of $H_0^1(\Omega;|x|^\alpha)$ and denote by $||\cdot||_{-1}$ the norm in $H^{-1}(\Omega;|x|^\alpha)$. For any $f \in H^{-1}(\Omega;|x|^\alpha)$, consider the perturbed problem

$$\begin{cases} -\text{div}\,(|x|^\alpha \nabla u) = |u|^{2_\alpha^*-2}u + f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We say that a function $u \in H_0^1(\Omega;|x|^\alpha)$ is a solution of problem (3) if u is a critical point of the energy functional

$$J(u) = \frac{1}{2}\int_\Omega |x|^\alpha |\nabla u|^2 dx - \frac{1}{2_\alpha^*}\int_\Omega |u|^{2_\alpha^*} dx - \int_\Omega f u\, dx.$$

We observe that the Caffarelli-Kohn-Nirenberg inequality ensures that J is well defined on the space $H_0^1(\Omega;|x|^\alpha)$. Moreover, by the continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^{2_\alpha^*}(\Omega)$, the functional J is Fréchet differentiable on $H_0^1(\Omega;|x|^\alpha)$.

Perturbations of critical semilinear boundary value problems on bounded domains were initially studied by Tarantello in [11]. Our purpose is to prove a corresponding multiplicity result for the degenerate problem (3). Notice that in our case, Ω will be unbounded . We first need some preliminaries. Set

$$s_\alpha^0(\Omega) = \lim_{r\to 0} S_\alpha(\Omega \cap B_r)$$

and

$$s_\alpha^\infty(\Omega) = \lim_{r\to\infty} S_\alpha(\Omega \setminus B_r).$$

These limits are well defined because the mappings $r \longmapsto S_\alpha(\Omega \cap B_r)$ and $r \longmapsto S_\alpha(\Omega \setminus B_r)$ are easily seen to be respectively non increasing and non decreasing.

Condition $\mathcal{C}$. *We say that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) satisfies Condition $\mathcal{C}$ provided that Ω is a cone in $\mathbb{R}^N$, or $\Omega = \mathbb{R}^N$, or*

$$S_\alpha(\Omega) < \min\{s_\alpha^0(\Omega), s_\alpha^\infty(\Omega)\}\,. \tag{4}$$

We recall that $\Omega \subset \mathbb{R}^N$ is a cone if Ω has Lipschitz boundary and if $\lambda x \in \Omega$ for every $\lambda > 0$ and $x \in \Omega$. If Ω is a cone then

$$S_\alpha(\Omega) = s_\alpha^0(\Omega) = s_\alpha^\infty(\Omega),$$

so equality holds in (4) (see [5, Lemma 3.9]). We also point out (see Caldiroli-Musina [5]) the following situations in which property (4) is fulfilled:

(i) $\Omega = \Omega_0 \cup \Omega_1$, where Ω_0 is a cone and Ω_1 is an open bounded set such that $0 \notin \overline{\Omega_1}$;
(ii) $\Omega = I \times \mathbb{R}^{N-1}$, where $I = \mathbb{R}$, or $I = (0, +\infty)$, or $I = (-\infty, 0)$, or I is bounded and $0 \notin \overline{I}$.

Denote by E_+ the positive cone of $E = H^{-1}(\Omega; |x|^\alpha)$. This means that $f \in E_+$ if and only if $f \neq 0$ and

$$\int_\Omega f u dx \geq 0,$$

for any $u \in H_0^1(\Omega; |x|^\alpha)$ such that $u \geq 0$ a.e. in Ω.

Our main result is the following

Theorem 1.1. *Assume that $\alpha \in (0, 2)$ and Ω satisfies Condition $\mathcal{C}$. Then, for each $g \in E_+$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, problem (3) with $f = \varepsilon g$ has at least two positive solutions.*

Remark 1.2. *a) In the previous theorem, ε_0 can be chosen uniformly for g in a compact subset of E_+.*
b) The existence of at least two solutions (not necessarily positive) when g belongs to E instead of E_+ is less clear. The sign condition can easily be weakened, but we think the general case should require some additional assumption.

2 The first solution

We first recall that if c is a real number, X is a Banach space and $F : X \to \mathbb{R}$ is a C^1-functional then F satisfies condition $(PS)_c$ if any sequence (u_n) in X such that $F(u_n) \to c$ and $||F'(u_n)||_{X^*} \to 0$ as $n \to \infty$, is relatively compact. It is obvious that if a Palais-Smale sequence converges strongly, then its limit is a critical point. Our first result shows that if a $(PS)_c$ sequence of J is weakly convergent then its limit is a solution of problem (3).

Lemma 2.1. *Let $(u_n) \subset H_0^1(\Omega; |x|^\alpha)$ be a $(PS)_c$ sequence of J, for some $c \in \mathbb{R}$. Assume that (u_n) converges weakly to some u_0. Then u_0 is a solution of problem (3).*

Proof. Consider an arbitrary function $\zeta \in C_0^\infty(\Omega)$ and set $\omega = \mathrm{supp}\,(\zeta)$. Obviously $J'(u_n) \to 0$ in $H_0^1(\Omega; |x|^\alpha)$ implies $\langle J'(u_n), \zeta\rangle \to 0$ as $n \to \infty$, that is

$$\lim_{n\to\infty} \left(\int_\omega |x|^\alpha \nabla u_n \cdot \nabla \zeta \, dx - \int_\omega |u_n|^{2_\alpha^*-2} u_n \zeta \, dx - \int_\omega f \zeta \, dx \right) = 0. \tag{5}$$

Since $u_n \rightharpoonup u_0$ in $H_0^1(\Omega; |x|^\alpha)$ it follows that

$$\lim_{n\to\infty} \int_\omega |x|^\alpha \nabla u_n \cdot \nabla \zeta \, dx = \int_\omega |x|^\alpha \nabla u_0 \cdot \nabla \zeta \, dx. \tag{6}$$

The boundedness of (u_n) in $H_0^1(\Omega; |x|^\alpha)$ and the Caffarelli-Kohn-Nirenberg inequality imply that $|u_n|^{2_\alpha^*-2}u_n$ is bounded in $L^{2_\alpha^*/(2_\alpha^*-1)}(\Omega; |x|^\alpha)$. Combining this with the convergence (up to a subsequence)

$$|u_n|^{2_\alpha^*-2}u_n \to |u_0|^{2_\alpha^*-2}u_0 \qquad \text{a.e. in } \Omega$$

we deduce that $|u_0|^{2_\alpha^*-2}u_0$ is the weak limit of the sequence $|u_n|^{2_\alpha^*-2}u_n$ in the space $L^{2_\alpha^*/(2_\alpha^*-1)}(\Omega; |x|^\alpha)$. So

$$\lim_{n\to\infty} \int_\omega |u_n|^{2_\alpha^*-2}u_n \zeta \, dx = \int_\omega |u_0|^{2_\alpha^*-2}u_0 \zeta \, dx. \tag{7}$$

From (5), (6) and (7) we deduce that

$$\int_\omega |x|^\alpha \nabla u_0 \cdot \nabla \zeta \, dx - \int_\omega |u_0|^{2_\alpha^*-2}u_0 \zeta \, dx - \int_\omega f\zeta \, dx = 0.$$

By density, this equality holds for any $\zeta \in H_0^1(\Omega; |x|^\alpha)$ which means that $J'(u_0) = 0$. □

Lemma 2.2. *There exists $\varepsilon_1 > 0$ such that problem (3) has at least one solution u_0 provided that $f \neq 0$ and $||f||_{-1} < \varepsilon_1$. Moreover, u_0 is positive if $f \in E_+$.*

Proof. The idea is to show that there exist $c_0 < 0$ and $R > 0$ such that J has the $(PS)_{c_0}$ property, where

$$c_0 = \inf\{J(u); u \in H_0^1(\Omega; |x|^\alpha) \text{ and } ||u|| \le R\}. \tag{8}$$

Then we prove that c_0 is achieved by some $u_0 \in H_0^1(\Omega; |x|^\alpha)$ and, furthermore, $J'(u_0) = 0$.

Applying the Caffarelli-Kohn-Nirenberg inequality we have

$$\begin{aligned}
&J(u) = \frac{1}{2}||u||^2 - \frac{1}{2_\alpha^*}\int_\Omega |u|^{2_\alpha^*}dx - \int_\Omega fudx \ge \\
&\frac{1}{2}||u||^2 - \frac{1}{2_\alpha^*}\int_\Omega |u|^{2_\alpha^*}dx - ||f||_{-1} \cdot ||u|| \ge \\
&\left(\frac{1}{2} - \frac{\varepsilon^2}{2}\right)||u||^2 - C\,||u||^{2_\alpha^*} - C_\varepsilon\,||f||_{-1}^2 .
\end{aligned}$$

Fixing $\varepsilon \in (0,1)$ we find $R > 0$, $\varepsilon_1 > 0$ and $\delta > 0$ such that $J(u) \ge \delta$ if $||u|| = R$ and $||f||_{-1} < \varepsilon_1$.

Let c_0 be defined in (8). Since $f \neq 0$, $c_0 < J(0) = 0$. The set

$$\overline{B}_R := \{u \in H_0^1(\Omega; |x|^\alpha); \|u\| \le R\}$$

becomes a complete metric space with respect to the distance

$$\text{dist}\,(u, v) = \|u - v\| \qquad \text{for any } u, v \in \overline{B}_R.$$

On the other hand, J is lower semi-continuous and bounded from below on $\overline{B}_R$. So, by Ekeland's variational principle [8, Theorem 1.1], for any positive integer n there exists u_n such that

$$c_0 \le J(u_n) \le c_0 + \frac{1}{n}, \tag{9}$$

and

$$J(w) \geq J(u_n) - \frac{1}{n}\|u_n - w\| \quad \text{for all } w \in \overline{B}_R. \tag{10}$$

We claim that $\|u_n\| < R$ for n large enough. Indeed, if $\|u_n\| = R$ for infinitely many n, we may assume, without loss of generality, that $\|u_n\| = R$ for all $n \geq 1$. It follows that $J(u_n) \geq \delta > 0$. Combining this with (9) and letting $n \to \infty$, we have $0 \geq c_0 \geq \delta > 0$ which is a contradiction.

We now prove that $||J'(u_n)||_{-1} \to 0$. Indeed, for any $u \in H_0^1(\Omega; |x|^\alpha)$ with $\|u\| = 1$, let $w_n = u_n + tu$. For a fixed n, we have $\|w_n\| \leq \|u_n\| + t < R$, where $t > 0$ is small enough. Using (10) we obtain

$$J(u_n + tu) \geq J(u_n) - \frac{t}{n}\|u\|$$

that is

$$\frac{J(u_n + tu) - J(u_n)}{t} \geq -\frac{1}{n}\|u\| = -\frac{1}{n}.$$

Letting $t \searrow 0$, we deduce that $\langle J'(u_n), u\rangle \geq -\dfrac{1}{n}$ and a similar argument for $t \nearrow 0$ produces $|\langle J'(u_n), u\rangle| \leq \dfrac{1}{n}$ for any $u \in H_0^1(\Omega; |x|^\alpha)$ with $\|u\| = 1$. So,

$$\|J'(u_n)\|_{-1} = \sup_{\|u\|=1} |\langle J'(u_n), u\rangle| \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

We have obtained the existence of a $(PS)_{c_0}$ sequence, i.e. a sequence $(u_n) \subset H_0^1(\Omega; |x|^\alpha)$ with

$$J(u_n) \to c_0 \qquad \text{and} \qquad ||J'(u_n)||_{-1} \to 0. \tag{11}$$

But $\|u_n\| \leq R$ shows that (u_n) converges weakly in $H_0^1(\Omega; |x|^\alpha)$, up to a subsequence. Therefore, by (11) and Lemma 2.1 we find that for some $u_0 \in H_0^1(\Omega; |x|^\alpha)$,

$$u_n \rightharpoonup u_0 \text{ in } H_0^1(\Omega; |x|^\alpha), \quad u_n \to u_0 \text{ a.e. in } \mathbb{R}^N \tag{12}$$

and

$$J'(u_0) = 0. \tag{13}$$

We now prove that $J(u_0) = c_0$. By (11) and (12) we have

$$o(1) = \langle J'(u_n), u_n\rangle = \int_\Omega |x|^\alpha |\nabla u_n|^2\, dx - \int_\Omega |u_n|^{2_\alpha^*}\, dx - \int_\Omega f u_n\, dx.$$

Therefore

$$J(u_n) = \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \int_\Omega |u_n|^{2_\alpha^*}\, dx - \left(1 - \frac{1}{2_\alpha^*}\right) \int_\Omega f u_n\, dx + o(1).$$

By (11), (12), (13) and Fatou's lemma we have

$$c_0 = \liminf_{n\to\infty} J(u_n) \geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \int_\Omega |x|^\alpha |u_0|^{2_\alpha^*}\, dx - \left(1 - \frac{1}{2_\alpha^*}\right) \int_\Omega f u_0\, dx = J(u_0).$$

Since $u_0 \in \overline{B}_R$, it follows that $J(u_0) = c_0$. If $f \in E_+$, u_0 can be replaced by $|u_0|$, and the proof is complete. □

3 A priori estimates for the second solution

Set
$$I(u) = \frac{1}{2}\int_\Omega |x|^\alpha |\nabla u|^2 dx - \frac{1}{2^*_\alpha}\int_\Omega |u|^{2^*_\alpha} dx$$
and denote
$$S = \{u \in H^1_0(\Omega; |x|^\alpha) \setminus \{0\};\ \langle I'(u), u\rangle = 0\}.$$
We first justify that $S \neq \emptyset$. Indeed, fix $u_0 \in H^1_0(\Omega; |x|^\alpha) \setminus \{0\}$ and set, for any $\lambda > 0$,
$$\Psi(\lambda) = \langle I'(\lambda u_0), \lambda u_0\rangle = \lambda^2 \int_\Omega |x|^\alpha |\nabla u_0|^2\, dx - \lambda^{2^*_\alpha}\int_\Omega |u_0|^{2^*_\alpha}\, dx\,.$$
Since $2^*_\alpha > 2$, it follows that $\Psi(\lambda) < 0$ for λ large enough and $\Psi(\lambda) > 0$ for λ sufficiently close to zero.

Hence there exists $\lambda_0 \in (0, \infty)$ such that $\Psi(\lambda_0) = 0$. This means that $\lambda_0 u_0 \in S$.

Lemma 3.1. *Let $I_\infty = \inf\{I(u);\ u \in S\}$. Then there exists $\bar u \in H^1_0(\Omega; |x|^\alpha)$ such that*
$$I_\infty = I(\bar u) = \sup_{t\ge 0} I(t\bar u). \tag{14}$$

Proof. We first claim that
$$I_\infty(u) = \sup_{t\ge 0} I(tu) \qquad \forall u \in S\,. \tag{15}$$
Indeed, for some fixed $\varphi \in H^1_0(\Omega; |x|^\alpha) \setminus \{0\}$, denote
$$f(t) = I(t\varphi) = \frac{t^2}{2}\int_\Omega |x|^\alpha |\nabla u|^2\, dx - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_\Omega |\varphi|^{2^*_\alpha}\, dx.$$
We have
$$f'(t) = t\int_\Omega |x|^\alpha |\nabla u|^2\, dx - t^{2^*_\alpha - 1}\int_\Omega |\varphi|^{2^*_\alpha}\, dx,$$
which vanishes for
$$t_0 = t_0(\varphi) = \left\{\frac{\displaystyle\int_\Omega |x|^\alpha |\nabla u|^2\, dx}{\displaystyle\int_\Omega |\varphi|^{2^*_\alpha}\, dx}\right\}^{\frac{1}{2^*_\alpha - 2}}.$$
Hence
$$f(t_0) = I(t_0\varphi) = \sup_{t\ge 0} I(t\varphi) = \frac{2-\alpha}{2N}\left\{\frac{\displaystyle\int_\Omega |x|^\alpha |\nabla u|^2\, dx}{\left(\displaystyle\int_\Omega |\varphi|^{2^*_\alpha}\, dx\right)^{\frac{N-2+\alpha}{N}}}\right\}^{\frac{N}{2-\alpha}}.$$
It follows that
$$\inf_{\varphi \in H^1_0(\Omega; |x|^\alpha)\setminus\{0\}} \sup_{t\ge 0} I(t\varphi) = \frac{2-\alpha}{2N}[S_\alpha(\Omega)]^{\frac{N}{2-\alpha}}. \tag{16}$$
We now easily observe that for every $u \in S$ we have $t_0(u) = 1$. So, by (16), we find (15).

By Caldiroli-Musina [5], Theorems 2.2 and 3.1, the minimum is achieved in (2) by some function $U \in H_0^1(\Omega;|x|^\alpha)$. We prove in what follows that the function $\bar{u} := [S_\alpha(\Omega)]^{1/(2^*_\alpha-2)}U$ satisfies (14). We first observe that $\bar{u} \in S$ and

$$I(\bar{u}) = \frac{2-\alpha}{2N}[S_\alpha(\Omega)]^{\frac{N}{2-\alpha}}\,. \tag{17}$$

So, by (15) and (17),

$$I_\infty = \inf_{u\in S} I(u) = \inf_{u\in S}\sup_{t\geq 0} I(tu) \geq \inf_{u\in H_0^1(\Omega;|x|^\alpha)\setminus\{0\}}\ \sup_{t\geq 0} I(tu) = \frac{2-\alpha}{2N}[S_\alpha(\Omega)]^{\frac{N}{2-\alpha}} = I(\bar{u}),$$

which concludes our proof. □

Lemma 3.2. *Assume (u_n) is a $(PS)_c$ sequence of J that converges weakly to u_0 in $H_0^1(\Omega;|x|^\alpha)$. Then either (u_n) converges strongly in $H_0^1(\Omega;|x|^\alpha)$, or $c \geq J(u_0)+I_\infty$.*

Proof. Since (u_n) is a $(PS)_c$ sequence and $u_n \rightharpoonup u_0$ in $H_0^1(\Omega;|x|^\alpha)$ we have

$$J(u_n) = c+o(1) \quad \text{and} \quad \langle J'(u_n),u_n\rangle = o(1)\,. \tag{18}$$

Set $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in $H_0^1(\Omega;|x|^\alpha)$ which implies

$$\begin{aligned}&\int_\Omega |x|^\alpha \nabla v_n\cdot\nabla u_0\,dx \to 0 \quad \text{as } n\to\infty\,,\\ &\int_\Omega f v_n\,dx \to 0 \quad \text{as } n\to\infty\,.\end{aligned}$$

We rewrite the above relations as

$$\begin{aligned}&\|u_n\|^2 = \|u_0\|^2+\|v_n\|^2+o(1)\,,\\ &J(v_n) = I(v_n)+o(1)\,.\end{aligned} \tag{19}$$

The Brezis-Lieb Lemma combined with the Caffarelli-Kohn-Nirenberg Inequality yield

$$\int_\Omega (|u_n|^{2^*_\alpha}-|v_n|^{2^*_\alpha})dx = \int_\Omega |u_0|^{2^*_\alpha}dx + o(1)\,. \tag{20}$$

From (18), (19), (20) and Lemma 2.1 we find

$$\begin{aligned}&o(1)+c = J(u_n) = J(u_0)+J(v_n)+o(1) = J(u_0)+I(v_n)+o(1)\,,\\ &o(1) = \langle J'(u_n),u_n\rangle = \langle J'(u_0),u_0\rangle + \langle J'(v_n),v_n\rangle + o(1) = \langle I'(v_n),v_n\rangle + o(1)\,.\end{aligned} \tag{21}$$

If $v_n \to 0$ in $H_0^1(\Omega;|x|^\alpha)$, then $u_n\to u_0$ in $H_0^1(\Omega;|x|^\alpha)$ and $J(u_0) = \lim_{n\to\infty} J(u_n) = c$.

If $v_n \not\to 0$ in $H_0^1(\Omega;|x|^\alpha)$, then combining this with the fact that $v_n \rightharpoonup 0$ in $H_0^1(\Omega;|x|^\alpha)$ we may assume that $\|v_n\|\to l>0$. Then, by (21),

$$c = J(u_0)+I(v_n)+o(1) \tag{22}$$

$$\mu_n = \langle I'(v_n),v_n\rangle = \int_\Omega |x|^\alpha|\nabla v_n|^2\,dx - \int_\Omega |v_n|^{2^*_\alpha}\,dx = \alpha_n-\beta_n, \tag{23}$$

where $\lim_{n\to\infty}\mu_n = 0$, $\alpha_n = \int_\Omega |x|^\alpha|\nabla v_n|^2\,dx \geq \|v_n\|^2$ and $\beta_n = \int_\Omega |v_n|^{2^*_\alpha}\,dx \geq 0$. In virtue of (22), it remains to show that $I(v_n)\geq I_\infty+o(1)$. For $t>0$, we have

$$\langle I'(tv_n),tv_n\rangle = t^2\int_\Omega |x|^\alpha|\nabla v_n|^2\,dx - t^{2^*_\alpha}\int_\Omega |v_n|^{2^*_\alpha}\,dx.$$

If we prove the existence of a sequence (t_n) with $t_n \to 1$ and $\langle I'(t_n v_n), t_n v_n\rangle = 0$ then

$$I(v_n) = I(t_n v_n) + \frac{1-t_n^2}{2}\alpha_n - \frac{1-t_n^{2^*_\alpha}}{2^*_\alpha}\|v_n\|^{2^*_\alpha}_{L^{2^*_\alpha}} = I(t_n v_n) + o(1) \geq I_\infty + o(1)$$

and the conclusion follows. To do this, let $t = 1+\delta$ with $\delta > 0$ small enough and using (23) we obtain

$$\begin{aligned}&\langle I'(tv_n), tv_n\rangle = (1+\delta)^2\alpha_n - (1+\delta)^{2^*_\alpha}\beta_n = (1+\delta)^2\alpha_n - (1+\delta)^{2^*_\alpha}(\alpha_n - \mu_n) = \\ &\alpha_n(2\delta - 2^*_\alpha\delta + o(\delta)) + (1+\delta)^{2^*_\alpha}\mu_n = \alpha_n(2-2^*_\alpha)\delta + \alpha_n o(\delta) + (1+\delta)^{2^*_\alpha}\mu_n.\end{aligned}$$

Since $\alpha_n \to \bar{l} \geq l^2 > 0$, $\lim_{n\to\infty}\mu_n = 0$ and $2^*_\alpha > 2$ then, for n large enough, we can define the sequence $\delta_n = \frac{2\,|\mu_n|}{\alpha_n(2^*_\alpha - 2)} > 0$ and $\delta_n \to 0$. Then

$$\langle I'((1+\delta_n)v_n), (1+\delta_n)v_n\rangle < 0 \qquad \langle I'((1-\delta_n)v_n), (1-\delta_n)v_n\rangle > 0\,. \tag{24}$$

From (24) we deduce the existence of $t_n \in (1-\delta_n, 1+\delta_n)$ such that

$$t_n \to 1 \quad \text{and} \quad \langle I'(t_n v_n), t_n v_n\rangle = 0.$$

This concludes our proof. $\square$

Fix $\bar{u} \in H^1_0(\Omega; |x|^\alpha)$ such that (14) holds. Since $2 < 2^*_\alpha$, there exists $t_0 > 0$ such that

$$\begin{aligned}I(t\bar{u}) < 0 &\quad \text{if } t \geq t_0\\ J(t\bar{u}) < 0 &\quad \text{if } t \geq t_0\,.\end{aligned}$$

Set

$$\mathcal{P} = \{\gamma \in C([0,1], H^1_0(\Omega; |x|^\alpha));\ \gamma(0) = 0, \gamma(1) = t_0\bar{u}\} \tag{25}$$

$$c_1 = \inf_{\gamma\in\mathcal{P}}\sup_{u\in\gamma} J(u). \tag{26}$$

In the next result c_0, resp. c_1, are those defined in (8), resp. (26).

Lemma 3.3. *Given $g \in E_+$, $\|g\|_{-1} = 1$, there exist $R > 0$ and $\varepsilon_2 = \varepsilon_2(R) > 0$ such that $c_1 < c_0 + I_\infty$, for all $f = \varepsilon g$ with $\varepsilon \leq \varepsilon_2$.*

Proof. We first remark that

$$I_\infty + c_0 > 0\,, \tag{27}$$

provided that ε_1 and R given in the proof of Lemma 2.2 are sufficiently small. Indeed, let u_0 be the solution obtained in Lemma 2.2. Then, by Cauchy-Schwarz,

$$\begin{aligned}c_0 &= \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right)\int_\Omega |x|^\alpha|\nabla u_0|^2dx - \left(1 - \frac{1}{2^*_\alpha}\right)\int_\Omega fu_0dx\\ &\geq \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right)\int_\Omega |x|^\alpha|\nabla u_0|^2dx - \left(1 - \frac{1}{2^*_\alpha}\right)\|f\|_{-1}\cdot\|u_0\|\,.\end{aligned} \tag{28}$$

Applying the inequality

$$\alpha\beta \leq \frac{\alpha^2}{2} + \frac{\beta^2}{2} \qquad \forall \alpha,\beta > 0$$

we find

$$\left(1-\frac{1}{2^*_\alpha}\right)\|f\|_{-1}\cdot\|u_0\| \le \left(\frac{1}{2}-\frac{1}{2^*_\alpha}\right)\|u_0\|^2+\frac{(N-\alpha+2)^2}{16N(2-\alpha)}\|f\|^2_{-1}\,. \tag{29}$$

So, by (28) and (29),

$$c_0 \ge -\frac{(N-\alpha+2)^2}{16N(2-\alpha)}\|f\|^2_{-1}\,. \tag{30}$$

It follows that the negative number c_0 is close enough to 0 if $\|f\|_{-1}$ is small. But, by Lemma 3.1,

$$I_\infty = \frac{2-\alpha}{2N}\,[S_\alpha(\Omega)]^{N/(2-\alpha)} > 0,$$

so (27) follows obviously.

In order to conclude the proof we observe, by the definition of c_1, that it suffices to show that

$$\sup_{t\ge 0} J(t\bar{u}) < c_0 + I_\infty\,, \tag{31}$$

if $\|f\|_{-1}$ is sufficiently small.

Next, using (27), the continuity of J and $J(0)=0$, we obtain some $T_0>0$ which is uniform with respect to all f satisfying $0<\|f\|_{-1}<\varepsilon_1$ such that, for some $\varepsilon'<\varepsilon_1$,

$$c_0 + I_\infty > \sup_{t\in[0,T_0]} J(t\bar{u})\,,$$

if $\|f\|_{-1}<\varepsilon'$. So, in order to prove (31), it suffices to show that if $\|f\|_{-1}$ is small then

$$c_0 + I_\infty > \sup_{t\ge T_0} J(t\bar{u})\,. \tag{32}$$

But

$$\begin{aligned} J(t\bar{u}) &= \frac{t^2}{2}\int_\Omega |x|^\alpha|\nabla\bar{u}|^2dx - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_\Omega|\bar{u}|^{2^*_\alpha}dx - t\int_\Omega f\bar{u}dx \\ &\le \frac{t^2}{2}\int_\Omega |x|^\alpha|\nabla\bar{u}|^2dx - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_\Omega|\bar{u}|^{2^*_\alpha}dx - T_0\int_\Omega f\bar{u}dx\,, \end{aligned}$$

for any $t\ge T_0$. But, by Lemma 3.1,

$$I(\bar{u}) = \frac{2-\alpha}{2N}\,[S_\alpha(\Omega)]^{N/(2-\alpha)}\,.$$

Hence, using an argument similar to that used for proving (28), we find

$$\begin{aligned} \sup_{t\ge T_0} J(t\bar{u}) &\le \sup_{t\ge T_0}\left(\frac{t^2}{2}\int_\Omega |x|^\alpha|\nabla\bar{u}|^2dx - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_\Omega|\bar{u}|^{2^*_\alpha}dx\right) - T_0\int_\Omega f\bar{u}dx \\ &\le I_\infty - T_0\int_\Omega f\bar{u}dx < I_\infty + c_0\,, \end{aligned}$$

if $f=\varepsilon g$ with $\varepsilon\le\varepsilon''$. Indeed, it follows from (30) that c_0 is quadratic in ε while $\int f\overline{u}$ is linear. Letting $\varepsilon_2=\min\{\varepsilon',\varepsilon''\}$, we conclude the proof. □

4 Proof of Theorem 1.1 concluded

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Hence, by Lemma 2.2, we obtain the existence of a positive solution $u_0 \in H_0^1(\Omega; |x|^\alpha)$ of (3) such that $J(u_0) = c_0$.

On the other hand, since $J(|u|) \le J(u)$ when $f \in E_+$, it follows from the Mountain Pass Theorem without the Palais-Smale condition [3, Theorem 2.2] that there exists a positive $(PS)_{c_1}$ sequence (u_n) of J, that is

$$J(u_n) = c_1 + o(1) \quad \text{and} \quad \|J'(u_n)\|_{-1} \to 0 \,.$$

This implies

$$\begin{aligned} c_1 + \frac{1}{2^*_\alpha}\|J'(u_n)\|_{-1} \cdot \|u_n\| + o(1) &\ge J(u_n) - \frac{1}{2^*_\alpha}\langle J'(u_n), u_n\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right)\|u_n\|^2 - \left(1 - \frac{1}{2^*_\alpha}\right)\|f\|_{-1} \cdot \|u_n\|. \end{aligned} \tag{33}$$

Hence $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega; |x|^\alpha)$. So, up to a subsequence, we may assume that $u_n \rightharpoonup u_1 \ge 0$ in $H_0^1(\Omega; |x|^\alpha)$. Lemma 2.1 implies that u_1 is a solution of (3).

We prove in what follows that $u_0 \ne u_1$. For this aim we shall prove that $J(u_0) \ne J(u_1)$. Indeed, by Lemma 3.2, either $u_n \to u_1$ in $H_0^1(\Omega; |x|^\alpha)$ which gives

$$J(u_1) = \lim_{n\to\infty} J(u_n) = c_1 > 0 > c_0 = J(u_0)$$

and the conclusion follows, or

$$c_1 = \lim_{n\to\infty} J(u_n) \ge J(u_1) + I_\infty.$$

If we suppose that $J(u_1) = J(u_0) = c_0$, then $c_1 \ge c_0 + I_\infty$ which contradicts Lemma 3.3. This concludes our proof. □

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Nonlinear eigenvalue problems for quasilinear operators on unbounded domains

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Abstract. We prove several existence results for eigenvalue problems involving the p-Laplacian and a nonlinear boundary condition on unbounded domains. We treat the non-degenerate subcritical case and the solutions are found in an appropriate weighted Sobolev space.

1 Introduction and preliminary results

The growing attention for the study of the p-Laplacian operator Δ_p in the last few decades is motivated by the fact that it arises in various applications. For instance, in Fluid Mechanics, the shear stress $\vec{\tau}$ and the velocity gradient $\nabla_p u$ of certain fluids obey a relation of the form $\vec{\tau}(x) = a(x)\nabla_p u(x)$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$. Here $p > 1$ is an arbitrary real number and the case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudoplastic, dilatant) fluid. The resulting equations of motion then involve $\operatorname{div}(a\nabla_p u)$, which reduces to $a\Delta_p u = a \operatorname{div}(\nabla_p u)$, provided that a is constant. The p-Laplacian appears in the study of flow through porous media ($p = 3/2$, see Showalter-Walkington [24]) or glacial sliding ($p \in (1, 4/3]$, see Pélissier-Reynaud [20]). We also refer to Aronsson-Janfalk [4] for the mathematical treatment of the Hele-Shaw flow of "power-law fluids". The concept of Hele-Shaw flow refers to the flow between two closely-spaced parallel plates, close in the sense that the gap between the plates is small compared to the dimension of the plates. Quasilinear problems with a variable coefficient also appear in the mathematical model of the torsional creep (elastic for $p = 2$, plastic as $p \to \infty$, see Bhattacharya-DiBenedetto-Manfredi [5] and Kawohl [18]). This study is based on the observation that a prismatic material rod subject to a torsional moment, at sufficiently high temperature and for an extended period of time, exhibits a permanent deformation, called *creep*. The corresponding equations are derived under the assumptions that the components of strain and stress are linked by a power law referred to as the *creep-law* see Kachanov [16, Chapters IV, VIII], Kachanov [17], and Findley-Lai-Onaran [13]). A nonlinear field equation in Quantum Mechanics involving the p-Laplacian, for $p = 6$, has been proposed in Benci-Fortunato-Pisani [6]. Eigenvalue problems involving the p-Laplacian have been the subject of much recent interest (we refer only to Allegretto-Huang [1], Anane [3], Drábek [9], Drábek-Pohozaev [11], Drábek-Simader [12], García-Peral [15], García-Montefusco-Peral [14]).

Let $\Omega \subset \mathbf{R}^N$ be an unbounded domain with (possible noncompact) smooth boundary $\partial\Omega$. We assume throughout this paper that p, q and m are real numbers satisfying $1 < p < q < p^* = \frac{Np}{N-p}$, if $p < N$ ($p^* = +\infty$ if $p \geq N$), $q \leq m < \frac{p(N-1)}{N-p}$ if $p < N$ ($q \leq m < +\infty$ when $p \geq N$).

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbf{R}^N)$-functions restricted on Ω.

We define the weighted Sobolev space E as the completion of $C_\delta^\infty(\Omega)$ in the norm

$$\|u\|_E = \left(\int_\Omega \left(|\nabla u(x)|^p + \frac{1}{(1+|x|)^p} |u(x)|^p \right) dx \right)^{1/p}.$$

Denote by $L^p(\Omega;\, w_1), L^q(\Omega;\, w_2)$ and $L^m(\partial\Omega;\, w_3)$ the weighted Lebesgue spaces with weight functions $w_i(x) = (1+|x|)^{\alpha_i}$ $(i = 1, 2, 3)$, and the norms defined by

$$\|u\|_{p,w_1}^p = \int_\Omega w_1 |u(x)|^p \, dx, \quad \|u\|_{q,w_2}^q = \int_\Omega w_2 |u(x)|^q \, dx$$

and

$$\|u\|_{m,w_3}^m = \int_{\partial\Omega} w_3 |u(x)|^m \, dS,$$

where $-N < \alpha_1 < -p$ if $p < N$ ($\alpha_1 < -p$ when $p \geq N$), $-N < \alpha_2 < q\frac{N-p}{p} - N$ if $p < N$ ($-N < \alpha_2 < 0$ when $p \geq N$), and $-N < \alpha_3 < m\frac{N-p}{p} - N + 1$ if $p < N$ ($-N < \alpha_3 < 0$ when $p \geq N$).

We shall use in our paper the following embedding result.

Theorem A. *Under the above assumptions on p, q and m, the space E is compactly embedded in $L^q(\Omega;\, w_2)$ and also in $L^m(\partial\Omega;\, w_3)$.*

This theorem is a consequence of Theorem 2 and Corollary 6 of Pflüger [22]. Furthermore, with the same proof as in Pflüger [21, Lemma 2], one can show

Lemma 1 *The quantity*

$$\|u\|_b^p = \int_\Omega a(x) |\nabla u|^p \, dx + \int_{\partial\Omega} b(x) |u|^p \, dS$$

defines an equivalent norm on E.

2 The main results

Consider the problem

$$(A) \qquad \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x)|u|^{p-2}u + g(x)|u|^{q-2}u & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = h(x,u) & \text{on } \partial\Omega\,, \end{cases}$$

where n denotes the unit outward normal on $\partial\Omega$, $0 < a_0 \leq a \in L^\infty(\Omega)$, while $b : \partial\Omega \to \mathbf{R}$ is a continuous function satisfying

$$\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}},$$

for some constants $0 < c \leq C$.

Problems of this type arise in the study of physical phenomena related to equilibrium of anisotropic continuous media which possible are somewhere "perfect" insulators, cf. Dautray-Lions [7].

We assume that f and g are nontrivial measurable functions satisfying

$$0 \leq f(x) \leq C(1+|x|)^{\alpha_1} \quad \text{and} \quad 0 \leq g(x) \leq C(1+|x|)^{\alpha_2}, \quad \text{for a.e. } x \in \Omega.$$

The mapping $h : \partial\Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function which fulfills the assumption

$(A1)$ $\quad |h(x,s)| \leq h_0(x) + h_1(x)|s|^{m-1}$,

where $h_i : \partial\Omega \to \mathbf{R}$ $(i = 0, 1)$ are measurable functions satisfying

$$h_0 \in L^{m/(m-1)}(\partial\Omega;\, w_3^{1/(1-m)}) \quad \text{and} \quad 0 \leq h_i \leq C_h w_3 \quad \text{a.e. on } \partial\Omega.$$

We also assume

$(A2)$ $\quad \displaystyle\lim_{s\to 0} \frac{h(x,s)}{b(x)|s|^{p-1}} = 0$ uniformly in x.

$(A3)$ $\quad$ There exists $\mu \in (p, q]$ such that

$$\mu H(x,t) \leq t h(x,t) \text{ for a.e. } x \in \partial\Omega \text{ and every } t \in \mathbf{R}.$$

$(A4)$ $\quad$ There is a nonempty open set $U \subset \partial\Omega$ with $H(x,t) > 0$ for $(x,t) \in U \times (0,\infty)$, where $H(x,t) = \displaystyle\int_0^t h(x,s)\,ds$.

Our first result asserts that under the above hypotheses, problem (A) has at least a solution.

By weak solution of problem (A) we mean a function $u \in E$ such that, for any $v \in E$,

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v\; dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv\; dS$$

$$= \lambda \int_\Omega f(x)|u|^{p-2}uvdx + \int_\Omega g(x)|u|^{q-2}uvdx + \int_{\partial\Omega} h(x,u)vdS.$$

Define

$$\tilde{\lambda} := \inf_{u\in E;\; u\neq 0} \left(\frac{\int_\Omega a(x)|\nabla u|^p\,dx + \int_{\partial\Omega} b(x)|u|^p\,dS}{\int_\Omega f(x)|u|^p\,dx} \right).$$

Our first result is

Theorem 1 *Assume that the conditions $(A1)$-$(A4)$ hold. Then, for every $\lambda < \tilde{\lambda}$, problem (A) has a nontrivial weak solution.*

In the special case $h(x,s) \equiv 0$ we are able to show also a multiplicity result for problem (A). The statement is the following

Theorem 2 *Assume $h(x,s) \equiv 0$ and $q \geq 2$. Then, for every $\lambda < \tilde{\lambda}$, problem (A) possesses infinitely many solutions.*

Next we prove the existence of an eigensolution to the following eigenvalue problem

$$(B) \qquad \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda \left(f(x)|u|^{p-2}u + g(x)|u|^{q-2}u \right) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = \lambda h(x,u) & \text{on } \partial\Omega. \end{cases}$$

We stress that for the next existence result of the paper we drop the assumptions $(A2)$ and $(A4)$. By weak solution of problem (B) we mean a function $u \in E$ such that, for any $v \in E$,

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v \; dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv \; dS$$
$$= \lambda \left[\int_\Omega f(x)|u|^{p-2}uvdx + \int_\Omega g(x)|u|^{q-2}uvdx + \int_{\partial\Omega} h(x,u)vdS \right].$$

We prove

Theorem 3 *Assume that the hypotheses* $(A1)$ *and* $(A3)$ *hold. Let* d *be an arbitrary real number such that* $1/d$ *is not an eigenvalue* λ *in problem* (B), *and satisfying*

$$d > \frac{1}{\tilde{\lambda}}. \tag{2.1}$$

Then there exists $\overline{\rho} > 0$ *such that for all* $r > \rho \geq \overline{\rho}$, *the eigenvalue problem* (B) *has an eigensolution* $(u, \lambda) = (u_d, \lambda_d) \in E \times \mathbf{R}$ *for which one has*

$$\lambda_d \in \left[\frac{1}{d + r^2 \|u_d\|_b^{m-p}}, \frac{1}{d + \rho^2 \|u_d\|_b^{m-p}} \right].$$

3 Problem (A)

Throughout this section we use the same notations as was previously done in the case of problem (A).

The energy functional corresponding to (A) is defined as $F : E \to \mathbf{R}$

$$F(u) = \frac{1}{p}\int_\Omega a(x)|\nabla u|^p \, dx + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p \, dS - \frac{\lambda}{p}\int_\Omega f(x)|u|^p \, dx - \int_{\partial\Omega} H(x,u) \, dS - \frac{1}{q}\int_\Omega g(x)|u|^q \, dx$$

where H denotes the primitive function of h with respect to the second variable.

By Lemma 1 we have $\|\cdot\|_b \simeq \|\cdot\|_E$. We may write

$$F(u) = \frac{1}{p}\|u\|_b^p - \frac{\lambda}{p}\int_\Omega f(x)|u|^p \, dx - \int_{\partial\Omega} H(x,u) \, dS - \frac{1}{q}\int_\Omega g(x)|u|^q \, dx.$$

Since $p < q < p^*$, $-N < \alpha_1 < -p$ and $-N < \alpha_2 < q\frac{N-p}{p} - N$ we can apply Theorem A and we obtain that the embeddings $E \subset L^p(\Omega; w_1)$ and $E \subset L^q(\Omega; w_2)$ are compact. So the functional F is well defined.

We denote by $N_h = h(x, u(x))$, $N_H = H(x, u(x))$ the corresponding Nemytskii operators.

Lemma 2 *The operators*

$$N_h : L^m(\partial\Omega; w_3) \to L^{m/(m-1)}(\partial\Omega; w_3^{1/(1-m)}), \quad N_H : L^m(\partial\Omega; w_3) \to L^1(\partial\Omega)$$

are bounded and continuous.

Proof. The proof follows from Theorem 1.1 in [10]. ■

Our hypothesis $\lambda < \tilde{\lambda}$ implies the existence of some $C_0 > 0$ such that, for every $v \in E$

$$\|v\|_b^p - \lambda \int_\Omega f(x)|v|^p dx \geq C_0 \|v\|_b^p.$$

Lemma 3 *Under assumptions (A1)-(A4), the functional F is Fréchet differentiable on E and satisfies the Palais-Smale condition.*

Proof. Denote $I(u) = \frac{1}{p}\|u\|_b^p$, $K_H(u) = \int_{\partial\Omega} H(x,u)\, dS$, $K_\Psi(u) = \int_\Omega \Psi(x,u)\, dx$ and $K_\Phi(u) = \int_\Omega \Phi(x,u)\, dx$, where $\Phi(x,u) = \frac{1}{p} f(x)|u|^p$ and $\Psi(x,u) = \frac{1}{q} g(x)|u|^q$.

Then the directional derivative of F in the direction $v \in E$ is

$$\langle F'(u), v\rangle = \langle I'(u), v\rangle - \lambda\langle K'_\Phi(u), v\rangle - \langle K'_\Psi(u), v\rangle - \langle K'_H(u), v\rangle,$$

where

$$\langle I'(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v\, dx + \int_{\partial\Omega} b(x)|u|^{p-2} uv\, dS,$$

$$\langle K'_H(u), v\rangle = \int_{\partial\Omega} h(x,u) v\, dS,$$

$$\langle K'_\Psi(u), v\rangle = \int_\Omega g(x)|u|^{q-2} uv\, dx,$$

$$\langle K'_\Phi(u), v\rangle = \int_\Omega f(x)|u|^{p-2} uv\, dx.$$

Clearly, $I' : E \to E^\star$ is continuous. The operator K'_H is a composition of the operators

$$K'_H : E \to L^m(\partial\Omega;\, w_3) \overset{N_h}{\to} L^{m/(m-1)}(\partial\Omega;\, w_3^{1/(1-m)}) \overset{l}{\to} E^\star$$

where $\langle l(u), v\rangle = \int_{\partial\Omega} uv\, dS$. Since

$$\int_{\partial\Omega} |uv|\, dS \leq \left(\int_{\partial\Omega} |u|^{m'} w_3^{1/(1-m)}\, dS\right)^{1/m'} \left(\int_{\partial\Omega} |v|^m w_3\, dS\right)^{1/m},$$

then l is continuous, by Theorem A. As a composition of continuous operators, K'_H is continuous, too. Moreover, by our assumptions on w_3, the trace operator $E \to L^m(\partial\Omega;\, w_3)$ is compact and therefore, K'_H is also compact.

Set $\varphi(u) = f(x)\,|u|^{p-2}u$. By the proof of Lemma 2 we deduce that the Nemytskii operator corresponding to any function which satisfies $(A1)$ is bounded and continuous. Hence N_h and N_φ are bounded and continuous. We note that

$$K'_\Phi : E \subset L^p(\Omega;\, w_1) \overset{N_\varphi}{\to} L^{p/(p-1)}(\Omega;\, w_1^{1/(1-p)}) \overset{\eta}{\to} E^\star$$

where $\langle \eta(u), v\rangle = \int_\Omega uv\, dx$. Since

$$\int_\Omega |uv|\, dx \leq \left(\int_\Omega |u|^{p/(p-1)} w_1^{1/(1-p)}\, dx\right)^{(p-1)/p} \left(\int_\Omega |v|^p w_1\, dx\right)^{1/p},$$

it follows that η is continuous. But K'_Φ is the composition of three continuous operators and by the assumptions on w_1, the embedding $E \subset L^p(\Omega;\, w_1)$ is compact. This implies that K'_Φ is compact. In a similar way we obtain that K'_Ψ is compact and the continuous Fréchet differentiability of F follows.

Now, let $u_n \in E$ be a Palais-Smale sequence, i.e.,

$$|F(u_n)| \leq C \text{ for all } n \tag{3.1}$$

and

$$\|F'(u_n)\|_{E^\star} \to 0 \text{ as } n \to \infty. \tag{3.2}$$

We first prove that $\{u_n\}$ is bounded in E. Remark that (3.2) implies that

$$|\langle F'(u_n), u_n\rangle| \leq \mu \cdot \|u_n\|_b \text{ for } n \text{ large enough.}$$

This and (3.1) imply

$$C + \|u_n\|_b \geq F(u_n) - \frac{1}{\mu}\langle F'(u_n), u_n\rangle. \tag{3.3}$$

But

$$\langle F'(u_n), u_n\rangle = \int_\Omega a(x)|\nabla u_n|^p\, dx + \int_{\partial\Omega} b(x)|u_n|^p\, dS - \lambda \int_\Omega f(x)|u_n|^p\, dx - \int_\Omega g(x)|u_n|^q\, dx - \int_{\partial\Omega} h(x,u_n)u_n\, dS.$$

We have

$$F(u_n) - \frac{1}{\mu}\langle F'(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{\mu}\right)\left(\|u_n\|_b^p - \lambda \int_\Omega f(x)|u|^p dx\right)$$

$$-\left(\int_{\partial\Omega} H(x,u_n)\, dS - \frac{1}{\mu}\int_{\partial\Omega} h(x,u_n)u_n\, dS\right) - \left(\frac{1}{q} - \frac{1}{\mu}\right)\int_\Omega g(x)|u_n|^q\, dx).$$

By $(A3)$ we deduce that

$$\int_{\partial\Omega} H(x,u_n)\, dS \leq \frac{1}{\mu}\int_{\partial\Omega} h(x,u_n)u_n\, dS. \tag{3.4}$$

Therefore

$$F(u_n) - \frac{1}{\mu}\langle F'(u_n), u_n\rangle \geq \left(\frac{1}{p} - \frac{1}{\mu}\right) C_0\|u_n\|_b^p. \tag{3.5}$$

Relations (3.3) and (3.5) yield

$$C + \|u_n\|_b \geq \left(\frac{1}{p} - \frac{1}{\mu}\right) C_0\|u_n\|_b^p.$$

This shows that $\{u_n\}$ is bounded in E.

To prove that $\{u_n\}$ contains a Cauchy sequence we use the following inequalities for $\xi, \zeta \in \mathbf{R}^N$ (see Diaz [8], Lemma 4.10):

$$|\xi - \zeta|^p \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \quad \text{for } p \geq 2 \tag{3.6}$$

$$|\xi - \zeta|^2 \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \quad \text{for } 1 < p < 2. \tag{3.7}$$

Then we obtain in the case $p \geq 2$:

$$
\begin{aligned}
\|u_n - u_k\|_b^p &= \int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx + \int_{\partial\Omega} b(x)|u_n - u_k|^p\,dS \\
&\leq C(\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle) \\
= C(\langle F'(u_n), u_n - u_k\rangle &- \langle F'(u_k), u_n - u_k\rangle + \lambda\langle K_\Phi'(u_n), u_n - u_k\rangle - \lambda\langle K_\Phi'(u_k), u_n - u_k\rangle \\
+\langle K_H'(u_n), u_n - u_k\rangle &- \langle K_H'(u_k), u_n - u_k\rangle + \langle K_\Psi'(u_n), u_n - u_k\rangle - \langle K_\Psi'(u_k), u_n - u_k\rangle) \\
&\leq C(\|F'(u_n)\|_{E^\star} + \|F'(u_k)\|_{E^\star} + |\lambda|\,\|K_\Phi'(u_n) - K_\Phi'(u_k)\|_{E^\star} \\
&+\|K_H'(u_n) - K_H'(u_k)\|_{E^\star} + \|K_\Psi'(u_n) - K_\Psi'(u_k)\|_{E^\star})\|u_n - u_k\|_b.
\end{aligned}
$$

Since $F'(u_n) \to 0$ and K_Φ', K_Ψ', K_H' are compact, we can assume, passing eventually to a subsequence, that $\{u_n\}$ converges in E.

If $1 < p < 2$, then we use the estimate

$$
\|u_n - u_k\|_b^2 \leq C'|\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle|(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}). \tag{3.8}
$$

Since $\|u_n\|_b$ is bounded, the same arguments lead to a convergent subsequence. In order to prove the estimate (3.8) we recall the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$
(x+y)^s \leq C_s(x^s + y^s) \qquad \text{for any } x, y \in (0, \infty). \tag{3.9}
$$

Then we obtain

$$
\begin{aligned}
\|u_n - u_k\|_b^2 &= \left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx + \int_{\partial\Omega} b(x)|u_n - u_k|^p\,dS\right)^{\frac{2}{p}} \\
&\leq C_p\left[\left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx\right)^{\frac{2}{p}} + \left(\int_{\partial\Omega} b(x)|u_n - u_k|^p\,dS\right)^{\frac{2}{p}}\right].
\end{aligned} \tag{3.10}
$$

Using (3.7), (3.9) and the Hölder inequality we find

$$
\begin{aligned}
\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx &= \int_\Omega a(x)(|\nabla u_n - \nabla u_k|^2)^{\frac{p}{2}}\,dx \\
\leq C\int_\Omega a(x)\left((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\right)^{\frac{p}{2}} &(|\nabla u_n| + |\nabla u_k|)^{\frac{p(2-p)}{2}}\,dx \\
= C\int_\Omega\left(a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\right)^{\frac{p}{2}} &(a(x)(|\nabla u_n| + |\nabla u_k|)^p)^{\frac{2-p}{2}}\,dx \\
\leq C\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}} &\left(\int_\Omega a(x)(|\nabla u_n| + |\nabla u_k|)^p\,dx\right)^{\frac{2-p}{2}} \\
\leq \tilde{C}_p\left(\int_\Omega a(x)|\nabla u_n|^p\,dx + \int_\Omega a(x)|\nabla u_k|^p\,dx\right)^{\frac{2-p}{2}} &\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}} \\
\leq \overline{C}_p\left[\left(\int_\Omega a(x)|\nabla u_n|^p\,dx\right)^{\frac{2-p}{2}} + \left(\int_\Omega a(x)|\nabla u_k|^p\,dx\right)^{\frac{2-p}{2}}\right]
\end{aligned}
$$

$$\times \left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}}$$
$$\leq \overline{C}_p \left[\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right]^{\frac{p}{2}} \left(\|u_n\|_b^{\frac{(2-p)p}{2}} + \|u_k\|_b^{\frac{(2-p)p}{2}}\right).$$

Using the last inequality and (3.9) we have the estimate

$$\begin{aligned}&\left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx\right)^{\frac{2}{p}}\\ &\leq C'_p \left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}).\end{aligned} \tag{3.11}$$

In a similar way we can obtain the estimate

$$\left(\int_{\partial\Omega} b(x)|u_n - u_k|^p\,dS\right)^{\frac{2}{p}} \leq C'_p \left(\int_{\partial\Omega} b(x)(|u_n|^{p-2}u_n - |u_k|^{p-2}u_k)(u_n - u_k)\,dx\right)(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}). \tag{3.12}$$

It is now easy to observe that inequalities (3.10), (3.11) and (3.12) imply the estimate (3.8). The proof of Lemma 3 is complete. ■

Proof of Theorem 1. We have to verify the geometric assumptions of the Mountain-Pass Theorem. We first show that there exist positive constants R and c_0 such that

$$F(u) \geq c_0, \qquad \text{for any } u \in E \text{ with } \|u\| = R. \tag{3.13}$$

By Theorem A we obtain some $A > 0$ such that

$$\|u\|_{q,w_2}^q \leq A\|u\|_b^q \quad \text{for all } u \in E.$$

This fact implies that

$$F(u) = \frac{1}{p}\left(\|u\|_b^p - \lambda\|u\|_{p,w_1}^p\right) - \frac{1}{q}\int_\Omega g(x)|u|^q\,dx - \int_{\partial\Omega} H(x,u)\,dS \geq \frac{C_0}{p}\|u\|_b^p - \frac{A}{q}\|u\|_b^q - \int_{\partial\Omega} H(x,u)\,dS.$$

By $(A1)$ and $(A2)$ we deduce that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\frac{1}{q}|g(x)||u|^q \leq \varepsilon b(x)|u|^p + C_\varepsilon w_3(x)|u|^m.$$

Consequently

$$F(u) \geq \frac{C_0}{p}\|u\|_b^p - \frac{A}{q}\|u\|_b^q - \int_{\partial\Omega}\left(\varepsilon b(x)|u|^p + C_\varepsilon w_3(x)|u|^m\right)dS \geq \frac{C_0}{p}\|u\|_b^p - \frac{A}{q}\|u\|_b^q - \varepsilon c_1\|u\|_b^p - C_\varepsilon C_2\|u\|_b^m.$$

For $\varepsilon > 0$ and $R > 0$ small enough, we deduce that for every $u \in E$ with $\|u\|_b = R$, $F(u) \geq c_0 > 0$, which yields (3.13).

We verify in what follows the second geometric assumption of the Mountain-Pass Theorem, namely

$$\exists v \in E \text{ with } \|v\| > R \text{ such that } F(v) < c_0. \tag{3.14}$$

Choose $\psi \in C_\delta^\infty(\Omega)$, $\psi \geq 0$, such that $\emptyset \neq \mathrm{supp}\psi \cap \partial\Omega \subset U$. From $\frac{1}{q}g(x)|u|^q \geq c_3 s^\mu - c_4$ on $U \times (0,\infty)$ and $(A1)$ we claim that

$$F(t\psi) = \frac{t^p}{p}\left(\|\psi\|_b^p - \lambda\|\psi\|_{p,w_1}^p\right) - \frac{1}{q}\int_\Omega g(x)|t\psi|^q\,dx - \int_{\partial\Omega} H(x,t\psi)\,dS$$

$$\leq \frac{t^p}{p}\left(\|\psi\|_b^p - \lambda\|\psi\|_{p,w_1}^p\right) - c_3 t^\mu \int_U \psi^\mu\,dS + c_4|U| - \frac{t^q}{q}\int_\Omega w_2\psi^q\,dx.$$

Since $q \geq \mu > p$, we obtain $F(t\psi) \to -\infty$ as $t \to \infty$. It follows that if $t > 0$ is large enough, $F(t\psi) < 0$, so $v = t\psi$ satisfies (3.14).

By the Ambrosetti-Rabinowitz Theorem, problem (A) has a nontrivial weak solution. ∎

Next we prove the second existence result about problem (A).

Proof of Theorem 2. In order to show the claim we want to apply a classical tool in critical point theory, precisely we will use the Ljusternik-Schnirelmann theory (see [23]). Consider the even functional

$$J(v) = \frac{1}{p}\int_\Omega a(x)|\nabla v|^p\,dx + \frac{1}{p}\int_{\partial\Omega} b(x)|v|^p\,dS - \frac{\lambda}{p}\int_\Omega f(x)|v|^p\,dx,$$

on the closed symmetric manifold

$$M = \{v \in E : \int_\Omega g(x)|v|^q = 1\}.$$

Note that M is only a C^1-manifold, since we have assumed $1 < p < q$. By our hypotheses on f, g, b and h (note that $(A1)$-$(A4)$ are easily satisfied), Lemma 3 and Theorem 5.3 in [25], we have that $J|_M$ possesses at least $\gamma(M)$ pairs of critical points (where $\gamma(M)$ stands for the genus of M).

Now we have to estimate $\gamma(M)$. Since $g \not\equiv 0$ there exists an open set $\omega \subset \Omega$ such that $g(x) \geq \delta > 0$ on ω. By the properties of the genus it follows that $\gamma(\omega) \geq \gamma(B)$, where B is the unit ball of $W_0^{1,p}(\omega) \subset E$, but it is well known that the genus of the unit ball of a infinite dimensional Banach space is infinity, so $\gamma(M) = \infty$. Hence there exists a sequence $\{v_n\} \subset E$, such that any v_n (and also $-v_n$) is a constrained critical point of J on M.

By the Lagrange multipliers rule we obtain that there exists a sequence $\{\lambda_n\} \subset \mathbf{R}$ such that

$$\int_\Omega a(x)|\nabla v_n|^p\,dx + \int_{\partial\Omega} b(x)|v_n|^p\,dS - \lambda\int_\Omega f(x)|v_n|^p\,dx = \lambda_n\int_\Omega g(x)|v_n|^q\,dx.$$

Since $v_n \in M$, using our assumption $\lambda < \tilde{\lambda}$ we find

$$\lambda_n = \|v_n\|_b^p - \lambda\int_\Omega f(x)|v_n|^p\,dx > 0,$$

so we can apply the usual scaling. Setting $u_n = \lambda_n^{1/(q-p)}v_n$, we have that u_n satisfies for any n the equation

$$\int_\Omega a(x)|\nabla u_n|^p\,dx + \int_{\partial\Omega} b(x)|u_n|^p\,dS = \lambda\int_\Omega f(x)|u_n|^p\,dx + \int_\Omega g(x)|u_n|^q\,dx,$$

so the claim is proved. ∎

4 Problem (B)

We start with the following auxiliary result.

Lemma 4 *Under assumption (A1), if $q \le m$, there exists a number $\overline{\rho} > 0$ such that for each $\rho \ge \overline{\rho}$ the function*

$$v \mapsto \frac{\rho^2}{m}\|v\|_b^m - \frac{1}{p}\|v\|_{p,w_1}^p - \frac{1}{q}\int_\Omega g(x)|v|^q\ dx - \int_{\partial\Omega} H(x,v)\ dS, \quad v \in E,$$

is bounded from below on E.

Proof. The growth condition for h implies

$$\left|\int_{\partial\Omega} H(x,v)\ dS\right| \le \int_{\partial\Omega}\left(h_0(x)|v| + \frac{1}{m}h_1(x)|v|^m\right) dS$$

$$\le \left(\int_{\partial\Omega} h_0^{\frac{m}{m-1}} w_3^{\frac{1}{1-m}}\, dS\right)^{\frac{m-1}{m}} \|v\|_{L^m(\partial\Omega;w_3)} + C_h\|v\|_{L^m(\partial\Omega;w_3)}^m \le C_0 + C\|v\|_b^m, \quad v \in E,$$

with constants C_0,$C > 0$. One obtains also that

$$\frac{1}{q}\left|\int_\Omega g(x)|u|^q\ dx\right| \le C_2\|v\|_b^q \le \overline{C}_0 + \overline{C}\|v\|_b^m, \quad v \in E,$$

with constants $\overline{C}_0$,$\overline{C} > 0$. Clearly, we can choose now the positive number $\overline{\rho}$ as desired. ∎

In view of Lemma 4 one can find numbers $b_0 > 0$ and $\alpha > 0$ such that

$$\frac{\overline{\rho}^2}{m}\|v\|_b^m + \frac{2}{m}b_0 - \frac{1}{p}\|v\|_{p,w_1}^p - \frac{1}{q}\int_\Omega g(x)|v|^q\ dx - \int_{\partial\Omega} H(x,v)\ dS \ge \alpha > 0, \quad v \in E. \tag{4.1}$$

With $b_0 > 0$ and $\overline{\rho} > 0$ as above we consider numbers $r > \rho \ge \overline{\rho}$ and a function $\beta \in C^1(\mathbf{R})$ such that

$$\beta(0) = \beta(r) = 0,\ \beta(\rho) = b_0, \tag{4.2}$$

$$\beta'(t) < 0 \iff t < 0 \text{ or } \rho < t < r, \tag{4.3}$$

$$\lim_{|t|\to+\infty} \beta(t) = +\infty. \tag{4.4}$$

Lemma 5 *Assume that conditions $(A1)$ and $(A3)$ are fulfilled. Then, for any $d > 0$ satisfying (3), the functional $J : E \times \mathbf{R} \to \mathbf{R}$ defined by*

$$J(v,t) = \frac{t^2}{m}\|v\|_b^m + \frac{2}{m}\beta(t) - \frac{1}{p}\int_\Omega f(x)|v|^p - \frac{1}{q}\int_\Omega g(x)|v|^q\ dx - \int_{\partial\Omega} H(x,v)\ dx + \frac{d}{p}\|v\|_b^p \tag{4.5}$$

is of class C^1 and satisfies the Palais-Smale condition.

Proof. The property of J to be continuously differentiable has been already justified in the proof of Theorem 1.

In order to check the Palais-Smale condition let the sequences $\{v_n\} \subset E$ and $\{t_n\} \subset \mathbf{R}$ satisfy

$$|J(v_n, t_n)| \leq M, \ \forall n \geq 1 \tag{4.6}$$

$$J'_v(v_n, t_n) = t_n^2 \, \|v_n\|_b^{m-p} I'(v_n) - K'_\Phi(v_n) - K'_H(v_n) - K'_\Psi(v_n) + dI'(v_n) \to 0, \tag{4.7}$$

$$J'_t(v_n, t_n) = \frac{2}{m} \, (t_n \|v_n\|_b^m + \beta'(t_n)) \to 0 \tag{4.8}$$

where I, K_Φ, K_H, K_Ψ have been introduced in the proof of Lemma 3.

From (4.1), (4.2), (4.5) and (4.6) we infer that

$$M \geq \frac{t_n^2}{m} \, \|v_n\|_b^m + \frac{2}{m}\beta(t_n) - \frac{1}{p}\|v_n\|_{p,w_1}^p - \frac{1}{q}\int_\Omega g(x)|v_n|^q \, dx - \int_{\partial\Omega} H(x, v_n) \, dx + \frac{d}{p}\|v_n\|_b^p$$

$$\geq \frac{t_n^2 - \rho^2}{m} \, \|v_n\|_b^m + \frac{2}{m}(\beta(t_n) - \beta(\rho)) + \frac{d}{p} \, \|v_n\|_b^p.$$

Condition (4.4) in conjunction with the inequality above yields the boundedness of $\{t_n\}$.

Let us check the boundedness of $\{v_n\}$ along a subsequence. Without loss of generality we may admit that $\{v_n\}$ is bounded away from 0. From (22) we deduce that the sequence $\{t_n\|v_n\|_b^m\}$ is bounded. Therefore it is sufficient to argue in the case where $t_n \to 0$. From (4.6) it turns out that

$$\frac{1}{p}\|v_n\|_{p,w_1}^p + \int_\Omega H(x, v_n)dx + \frac{1}{q}\int_{\partial\Omega} g(x)|v_n|^q dx - \frac{d}{p}\|v_n\|_b^p$$

is bounded. By (4.7) we deduce that

$$\frac{1}{\|v_n\|_b}(-\langle K'_\Phi(v_n), v_n\rangle - \langle K'_H(v_n), v_n\rangle - \langle K'_\Psi(v_n), v_n\rangle + d\|v_n\|_b^p) \to 0.$$

Then, for n sufficiently large, assumption $(A3)$ allows us to write

$$M + 1 + \|v_n\|_b \geq d\left(\frac{1}{p} - \frac{1}{\mu}\right) \|v_n\|_b^p + \left(\frac{1}{\mu} - \frac{1}{q}\right) \|v_n\|_{L^q(\Omega, w_2)}^q$$

$$+ \int_{\partial\Omega} \left(\frac{1}{\mu}h(x, v_n)v_n - H(x, v_n)\right) \, dS + \left(\frac{1}{\mu} - \frac{1}{p}\right) \|v_n\|_{p,w_1}^p$$

$$\geq \left(\frac{1}{p} - \frac{1}{\mu}\right) \left(d\|v_n\|_b^p - \|v_n\|_{p,w_1}^p\right) \geq \left(\frac{1}{p} - \frac{1}{\mu}\right) \left(d - \frac{1}{\tilde{\lambda}}\right) \|v_n\|_b^p.$$

By (3), this establishes the boundedness of $\{v_n\}$ in E.

In view of the compactness of the mappings K'_Φ, K'_H, K'_Ψ (see the proof of Lemma 3), by (4.7) we get that

$$\left(d + t_n^2 \, \|v_n\|_b^{m-p}\right) \, I'(v_n)$$

converges in E as $n \to \infty$. The boundedness of $\{t_n\}$ and $\{v_n\}$ ensures that $\{I'(v_n)\}$ is convergent in E^* along a subsequence. Assume that $p \geq 2$. Inequality (3.6) shows that

$$\|u_n - u_k\|_b^p \leq C\left[\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k) \cdot (\nabla u_n - \nabla u_k) \, dx + \right.$$

$$+\int_\Gamma b(x)(|u_n|^{p-2}u_n - |u_k|^{p-2}u_k)(u_n-u_k)\,d\Gamma\Big] =$$

$$C\langle I'(u_n)-I'(u_k), u_n-u_k\rangle \le C\|I'(u_n)-I'(u_k)\|_b^*\|u_n-u_k\|_b \quad \text{if } p\ge 2.$$

Consequently, if $p\ge 2$, $\{v_n\}$ possesses a convergent subsequence. Proceeding in the same way with inequality (3.7) in place of (3.6) we obtain the result for $1<p<2$. ∎

In the proof of Theorem 3 we shall make use of the following variant of the Mountain Pass Theorem (see Motreanu [19])

Lemma 6 *Let E be a Banach space and let $J : E\times\mathbf{R}\to\mathbf{R}$ be a C^1 functional verifying the hypotheses*
(a) there exist constants $\rho>0$ and $\alpha>0$ such that $J(v,\rho)\ge\alpha$, for every $v\in E$;
(b) there is some $r>\rho$ with $J(0,0)=J(0,r)=0$.

Then the number

$$c := \inf_{g\in\mathcal{P}}\max_{0\le\tau\le1} J(h(\tau))$$

is a critical value of J, where

$$\mathcal{P} := \{g\in C([0,1];E\times\mathbf{R});\ g(0)=(0,0),\ g(1)=(0,r)\}.$$

Proof of Theorem 3. We apply Lemma 6 to the function J defined in (4.5). It is clear that assertion (a) is verified with $\rho>0$ and $\alpha>0$ described in Lemma 4 and (4.1). Due to relation (4.2), condition (b) in Lemma 6 holds. Lemma 5 ensures that the functional J satisfies the Palais-Smale condition. Therefore Lemma 6 yields a nonzero element $(u,t)\in E\times\mathbf{R}$ such that

$$J_v'(u,t) = \left(d+t^2\|u\|_b^{m-p}\right) I'(u) - K_\Phi'(u) - K_H'(u) - K_\Psi'(u) = 0, \tag{4.9}$$

$$J_t'(u,t) = \frac{2}{m}\ (t\|u\|_b^m + \beta'(t)) = 0. \tag{4.10}$$

From (4.10) it follows that

$$t\beta'(t)\le 0. \tag{4.11}$$

Combining (4.11) and (4.3) we derive that if $t\ne0$, then $u\ne0$ and

$$\rho\le t\le r. \tag{4.12}$$

Therefore for each d in (3) such that $1/d$ is not an eigenvalue in (B) and each $r>\rho\ge\overline{\rho}$ we deduce that there exists a critical point $(u,t)=(u_d,t_d)\in E\times\mathbf{R}_+$ of J, where $t=t_d$ verifies (4.12). Consequently, relation (4.9) establishes that $u_d\in E$ is an eigenfunction in problem (B) where the corresponding eigenvalue is

$$\lambda_d = \frac{1}{d+t_d^2\,\|u_d\|_b^{m-p}},$$

with $t=t_d$ satisfying (4.12). This completes the proof. ∎

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On a double bifurcation quasilinear problem arising in the study of anisotropic continuous media

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Abstract. We study the bifurcation problem

$$\begin{cases} -\mathrm{div}\,(a(x)|Du|^{p-2}Du) + h(x)u^{r-1} = f(\lambda, x, u) & \text{in } \Omega \subset \mathbf{R}^N\,, \\ a(x)|Du|^{p-2}Du \cdot n + b(x)u^{p-1} = \theta g(x,u) & \text{on } \Gamma\,, \\ u \geq 0, \quad u \not\equiv 0 & \text{in } \Omega\,, \end{cases}$$

where Ω is an unbounded domain with smooth non-compact boundary Γ, n denotes the unit outward normal vector on Γ, and $\lambda > 0$, θ are real parameters. We assume that $\max\{p, 2\} < r < p^* = pN/(N-p)$, $1 < p < N$, the functions a, b and h are positive while f, g are subcritical non-linearities. We show that there exist an open interval I and $\lambda^* > 0$ such that the problem has no solution if $\theta \in I$ and $\lambda \in (0, \lambda^*)$. Furthermore, there exist an open interval $J \subset I$ and $\lambda_0 > 0$ such that, for any $\theta \in J$, the above problem has at least a solution if $\lambda \geq \lambda_0$, but it has no solution provided that $\lambda \in (0, \lambda_0)$.
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1 Introduction

Among the great range of processes modelled by nonlinear equations, those leading to bifurcation problems are of particular difficulty and importance. More precisely, many models from chemical engineering, mathematical biology, mechanics and engineering may be written in the form

$$u_t = \mathcal{F}(\lambda, u, Du, D^2u, \cdots) \qquad \text{in } \Omega \times (0,T), \tag{1}$$

where $u = u(x,t)$ is the state of the system under consideration. For instance, if we try to describe the behaviour of a bacteria culture, then the state variable u might be the number of mass of the bacteria. In many concrete situations problems like (1) represent a complicated system of equations involving partial differential equations and other operations, like boundary or initial conditions. Each mathematical model contains (implicitly or explicitly) parameters corresponding to the real world situation being described. For example, the outcome of a bacteria growing experiment will depend on the size of the experimental apparatus, the temperature, the composition of the ambient atmosphere, and other parameters. In such a way, a surprising variety of the problems in applied mathematics which exhibit multiple steady state solutions, even systems with infinitely many degrees of freedom, can be reduced to the form

$$u_t = \mathcal{F}(\lambda_1, \lambda_2, \cdots, \lambda_k, u, Du, D^2u, \cdots) \qquad \text{in } \Omega \times (0,T)$$

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which involves a large number k of parameters. However, even for the biologists, it would be difficult to figure out how $\mathcal{F}$ should depend on all these quantities. In this case, in order to develop a consistent mathematical theory, one tries to fix as many as possible parameters and perhaps to vary one of them so as to see the effect of this. Many times several parameters in a model can be lumped into a single one by standard scaling procedures, such that Reynold's number, Lyapunov-Schmidt reduction, etc. Thus we obtain the evolution problem (1) which depends on a single parameter. The simplest solutions (1) can have are *equilibrium solutions.* These are time-independent solutions of (1), i.e., the states which satisfy $\mathcal{F}(\lambda, u, Du, D^2u, \cdots) = 0$. Similar problems arise for the case of several state variables. We refer, e.g., to the steady state Brusselator model (see Brown-Davidson [4]) which was developed to describe morphogenesis and pattern formation in chemical reactions. We assume in this paper that $\mathcal{F}$ involves the quasilinear differential operator

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du), \qquad 1 < p < \infty.$$

We are concerned in this paper with the study of the following double bifurcation quasilinear problem

$$(\mathrm{P}_{\lambda,\theta}) \begin{cases} -\operatorname{div}(a(x)|Du|^{p-2}Du) + h(x)u^{r-1} = f(\lambda, x, u) & \text{in } \Omega \subset \mathbf{R}^N, \\ a(x)|Du|^{p-2}Du \cdot n + b(x)u^{p-1} = \theta g(x,u) & \text{on } \Gamma, \\ u \geq 0, \quad u \not\equiv 0 & \text{in } \Omega, \end{cases}$$

where Ω is an unbounded domain with non-compact, smooth boundary Γ, $\lambda > 0$, θ are real parameters and throughout $\max\{p, 2\} < r < pN/(N-p)$, $1 < p < N$.

The study of non-trivial solutions in the above problem is motivated by the following example. Suppose an inviscid fluid flows irrotationally along a flat-bottomed canal. The flow can be modelled by an equation of the form $\mathcal{F}(\lambda, u, Du) = 0$, where $\mathcal{F}(\lambda, 0, 0) = 0$. One possible motion is a uniform stream (corresponding to the trivial solution $u = 0$), but it is of course the non-trivial solutions which are of physical interest.

Problems of this type arise in the study of physical phenomena related to equilibrium of anisotropic continuous media which possible are somewhere "perfect" insulators, cf. Dautray-Lions [7]. For instance, if $\vec{\tau}$ denotes the shear stress and $D_p u$ is the velocity gradient then these quantities obey a relation of the form $\vec{\tau}(x) = a(x)D_p u(x)$, where $D_p u = |Du|^{p-2}Du$. The case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudoplastic, dilatant) fluid. The resulting equations of motion then involve the quasilinear operator $\operatorname{div}(aD_p u)$. We refer in this sense to Aronsson-Janfalk [2] for the mathematical treatment of the Hele-Shaw flow of "power-law fluids". The concept of Hele-Shaw flow refers to the flow between two closely-spaced parallel plates, close in the sense that the gap between the plates is small compared to the dimension of the plates. Quasilinear problems with a variable coefficient also appear in the mathematical model of the torsional creep (elastic for $p = 2$, plastic as $p \to \infty$, see Bhattacharya-DiBenedetto-Manfredi [3] and Kawohl [14]). This study is based on the observation that a prismatic material rod subject to a torsional moment, at sufficiently high temperature and for an extended period of time, exhibits a permanent deformation, called *creep.* The corresponding equations are derived under the assumptions that the components of strain and stress are linked by a power law referred to as the *creep-law* see Kachanov [12, Chapters IV, VIII], Kachanov [13], and Findley-Lai-Onaran [11]). We also refer to the study of flow through porous media ($p = 3/2$, see Showalter-Walkington [19]) or glacial sliding ($p \in (1, 4/3]$, see Pélissier-Reynaud [15]). We mention the recent papers Cîrstea-Motreanu-Rădulescu [5], Drábek-Huang [9] and Drábek-Simader [10] for the mathematical treatment of bifurcation problems for several classes of quasilinear elliptic equations on unbounded domains and with respect to anisotropic spaces.

The purpose of this paper is to study a quasilinear eigenvalue problem with non-linear boundary condition in an unbounded domain $\Omega \subset \mathbf{R}^N$ and we generalize in a larger framework some results from Cîrstea-Rădulescu [6]. It is known that for unbounded domains, neither the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, nor the trace $W^{1,p}(\Omega) \to L^m(\Gamma)$ are compact. So, it is natural to look for more general function spaces, for instance weighted Sobolev spaces, where compact embeddings can be obtained for suitable weight functions. However, due to the non-linear boundary condition it is not only necessary to have compact embeddings of weighted Sobolev spaces but to use also compactness of the trace operator.

Pflüger [17] studied the trace operators $W^{1,p}(\Omega;\, v_0, v_1) \to L^s(\Gamma;\, w)$ in weighted Sobolev spaces for sufficiently regular unbounded domains $\Omega \subset \mathbf{R}^N$ with non-compact boundary. He established certain conditions on the weight functions v_0, v_1, w which ensures the compactness of this operator.

For a positive measurable function w_1 defined in a domain $\Omega \subset \mathbf{R}^N$, let $L^q(\Omega;\, w_1)$ be the space of all measurable functions u such that

$$\|u\|_{q,\Omega,w_1} = \left(\int_\Omega |u(x)|^q w_1(x)\, dx \right)^{1/q}$$

is finite. If Γ is a submanifold in $\mathbf{R}^N$, we denote by $L^m(\Gamma;\, w_2)$ the space of all measurable functions u such that $\|u\|_{m,\Gamma,w_2}$ is finite. The weighted Sobolev space $W^{1,p}(\Omega;\, v_0, v_1)$ is defined as the set of all functions $u \in L^p(\Omega;\, v_0)$ such that all the derivatives u_{x_i} $(1 \le i \le N)$ belong to $L^p(\Omega;\, v_1)$. The corresponding norm is given by

$$\|u\|_{1,p,\Omega,v_0,v_1} = \left(\int_\Omega |u(x)|^p v_0(x)\, dx + \int_\Omega |Du(x)|^p v_1(x)\, dx \right)^{1/p}.$$

Denote by A_p the Muckenhoupt class which is the set of all positive measurable functions v in $\mathbf{R}^N$ satisfying

$$\frac{1}{|Q|} \left(\int_Q v\, dx \right)^{1/p} \left(\int_Q v^{-1/(p-1)}\, dx \right)^{(p-1)/p} \le C \qquad \text{if } 1 < p < \infty$$

$$\frac{1}{|Q|} \int_Q v\, dx \le C \operatorname*{ess\,inf}_{x \in Q} v(x) \qquad \text{if } p = 1,$$

for all cubes Q in $\mathbf{R}^N$. For example, the function $v(x) = (1 + |x|)^\beta$ belongs to A_p if $\beta \in (-N, N(p-1))$ (see Torchinski [20]).

We always assume that the continuous weight functions v_0, v_1, w_0, w_1, w_2 belong to A_p. Furthermore, the unbounded domain $\Omega \subset \mathbf{R}^N$ and the weight functions are chosen such that we can apply [17, Theorem 2] and [17, Corollary 6] to guarantee that the trace $W^{1,p}(\Omega;\, v_0, v_1) \to L^p(\Gamma;\, w_0)$ is continuous and the embedding $W^{1,p}(\Omega;\, v_0, v_1) \hookrightarrow L^q(\Omega;\, w_1)$ for some $p < q < \frac{pN}{N-p}$, respectively the trace $W^{1,p}(\Omega;\, v_0, v_1) \to L^m(\Gamma;\, w_2)$ for some $p < m < p\,\frac{N-1}{N-p}$ are compact.

Remark 1 *To give an example of the domain $\Omega \subset \mathbf{R}^N$ and of the weight functions v_0, v_1, w_0, w_1 and w_2 that satisfy the above assumptions, consider Ω as an infinite cylinder $\omega \times \mathbf{R}$ where $\omega \subset \mathbf{R}^{N-1}$ is smooth, bounded and*

$$v_0(x) = \frac{1}{(1+|x|)^p}, \;\; v_1(x) = 1, \;\; w_0(x) = (1+|x|)^{\alpha_0}, \;\; w_1(x) = (1+|x|)^{\alpha_1}, \;\; w_2(x) = (1+|x|)^{\alpha_2}, \;\; x \in \mathbf{R}^N .$$

To obtain continuity of the trace operator $W^{1,p}(\Omega;\, v_0, v_1) \to L^p(\Gamma;\, w_0)$ and compactness of the embedding $W^{1,p}(\Omega;\, v_0, v_1) \hookrightarrow L^q(\Omega;\, w_1)$ respectively of the trace operator $W^{1,p}(\Omega;\, v_0, v_1) \to L^m(\Gamma;\, w_2)$ we have to choose

$$-N < \alpha_0 \leq 1-p, \quad -N < \alpha_1 < q\,\frac{N-p}{p} - N \quad \textit{and} \quad -N < \alpha_2 < m\,\frac{N-p}{p} - N + 1.$$

Denote by $C_\delta^\infty(\Omega)$ the space of $C_0^\infty(\mathbf{R}^N)$-functions restricted to Ω. We define the weighted Sobolev space E as the completion of $C_\delta^\infty(\Omega)$ in the norm $\|\cdot\|_E$ where we shall use the abbreviation $\|\cdot\|_E = \|\cdot\|_{1,p,\Omega,v_0,v_1}$.

Remark 2 *The definition of E and the choice of our weight functions ensure the continuity of the trace $E \to L^p(\Gamma;\, w_0)$ and the compactness of the embedding $E \hookrightarrow L^q(\Omega;\, w_1)$ respectively of the trace operator $E \to L^m(\Gamma;\, w_2)$.*

2 Main results

Suppose throughout this paper that the following hypotheses are fulfilled

(H$_1$) $v_0 \in C^1(\mathbf{R}^N)$ and there exists a constant $0 < \sigma < N$ such that

$$|x| \cdot |Dv_0(x)| \leq \sigma\, v_0(x) \quad \forall x \in \Omega\,;$$

(H$_2$) a is a positive measurable function, locally bounded in Ω and there exist positive constants a_0, a_1 such that

$$a_0(|x|^p v_0(x) + v_1(x)) \leq a(x) \leq a_1 v_1(x) \quad \text{a.e. } x \in \Omega\,;$$

(H$_3$) b is a positive continuous function on $\mathbf{R}^N$ and there exist positive constants b_0 and b_1 such that

$$b_0|x|v_0(x) \leq b(x) \leq b_1 w_0(x) \quad \text{a.e. } x \in \Gamma\,.$$

Let $f(\lambda, x, s) : (0,\infty) \times \Omega \times \mathbf{R} \to \mathbf{R}$ be non-decreasing in λ, measurable in x, derivable in s satisfying

(H$_4$) $f(\cdot,\cdot,0) = 0, \quad f(\lambda,x,s) + f(\lambda,x,-s) \geq 0 \quad \forall \lambda > 0$, a.e. $x \in \Omega$, $\forall s \in \mathbf{R}$;

(H$_5$) $|f_s(\lambda,x,s)| \leq \lambda\varphi(x)|s|^{q-2}$ for some $r > q > \max\{p, 2\}$, $\forall \lambda > 0$, a.e. $x \in \Omega$, $\forall s \in \mathbf{R}$, where φ is a non-negative, measurable function such that

$$0 \leq \varphi(x) \leq c_f\, w_1(x) \quad \text{a.e. } x \in \Omega\,;$$

(H$_6$) $\displaystyle\lim_{s\to 0} \frac{f(\lambda,x,s)}{\lambda w_1(x)|s|^{q-2}s} = 1$ uniformly in x and in λ;

(H$_7$) $|f(\lambda_1,x,s) - f(\lambda_2,x,s)| \leq |\lambda_1 - \lambda_2|\psi(x)|s|^{q-1}, \quad \forall \lambda_1, \lambda_2 > 0$, a.e. $x \in \Omega$, $\forall s \in \mathbf{R}$, where ψ is a non-negative, measurable function such that

$$0 \leq \psi(x) \leq C_f\, w_1(x) \quad \text{a.e. } x \in \Omega\,.$$

Assume $g : \Gamma \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function that satisfies the conditions

$(\mathbf{H_8})$ $\quad g(\cdot, 0) = 0, \quad g(x,s) + g(x,-s) \geq 0$ a.e. $x \in \Gamma$, $\forall s \in \mathbf{R}$;

$(\mathbf{H_9})$ $\quad |g(x,s)| \leq g_0(x) + g_1(x)|s|^{m-1}$, for some $p < m < p\,\frac{N-1}{N-p}$, a.e. $x \in \Omega$, $\forall s \in \mathbf{R}$, where g_0, g_1 are non-negative, measurable functions such that

$$0 \leq g_0(x), g_1(x) \leq C_g\, w_2(x) \quad \text{a.e. } x \in \Gamma\,, \quad g_0 \in L^{m/(m-1)}(\Gamma;\, w_2^{1/(1-m)}).$$

The following integrability condition of the ratio w_1^r/h^q is inspired by assumption (1.4) in Alama-Tarantello [1].

$(\mathbf{H_{10}})$ $\quad h : \Omega \to \mathbf{R}$ is a positive and continuous function satisfying

$$\int_\Omega \left(\frac{w_1^r}{h^q}\right)^{1/(r-q)} dx < \infty\,.$$

Remark 3 *If* $0 < \underline{a} \leq a \in L^\infty(\Omega)$ *and* $b \in C(\mathbf{R}^N)$ *is a positive function such that*

$$\frac{c_1}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{c_2}{(1+|x|)^{p-1}} \qquad for\ some\ constants\ 0 < c_1 \leq c_2$$

then hypotheses $(\mathbf{H_1})$*-*$(\mathbf{H_3})$ *are fulfilled if we take weight functions as in Remark 1 with* $\alpha_0 = 1-p$.

Consider the Banach space $X = E \cap L^r(\Omega;\, h)$ endowed with the norm

$$\|u\|_X^p := \|u\|_E^p + \left(\int_\Omega |u(x)|^r h(x)\,dx\right)^{p/r}.$$

Obviously, the following embeddings

$$X \overset{i}{\hookrightarrow} E \quad \text{and} \quad X \overset{j}{\hookrightarrow} L^r(\Omega;\, h) \quad \text{are continuous.} \tag{2}$$

The energy functional corresponding to $(P_{\lambda,\theta})$ is given by $\Phi_{\lambda,\theta} : X \to \mathbf{R}$,

$$\Phi_{\lambda,\theta}(u) = \frac{1}{p}\int_\Omega a(x)|Du|^p\,dx + \frac{1}{p}\int_\Gamma b(x)|u|^p\,d\Gamma + \frac{1}{r}\int_\Omega h(x)|u|^r\,dx - \int_\Omega F(\lambda,x,u)\,dx - \theta\int_\Gamma G(x,u)\,d\Gamma,$$

where F and G denote the primitive functions of f and g with respect to the last variable, i.e. $F(\lambda,x,u) = \int_0^u f(\lambda,x,s)\,ds$, $G(x,u) = \int_0^u g(x,s)\,ds$. Solutions to problem $(P_{\lambda,\theta})$ will be found as non-negative and non-trivial critical points of $\Phi_{\lambda,\theta}$. Therefore, a function $u \in X$ is a solution of the problem $(P_{\lambda,\theta})$ provided that $u \geq 0$, $u \not\equiv 0$ in Ω and for any $v \in X$,

$$\int_\Omega a(x)|Du|^{p-2}Du\cdot Dv\,dx + \int_\Gamma b(x)|u|^{p-2}uv\,d\Gamma + \int_\Omega h(x)|u|^{r-2}uv\,dx - \theta\int_\Gamma g(x,u)v\,d\Gamma = \int_\Omega f(\lambda,x,u)v\,dx.$$

Set

$$\mathcal{N}_g := \{u \in X : \int_\Gamma g(x,u)u\,d\Gamma < 0\}, \qquad \mathcal{P}_g := \{u \in X : \int_\Gamma g(x,u)u\,d\Gamma > 0\}$$

$$\theta_* := \sup_{u\in\mathcal{N}_g} \frac{\|u\|_b^p}{\int\limits_\Gamma g(x,u)u\,d\Gamma}, \qquad \theta^* := \inf_{u\in\mathcal{P}_g} \frac{\|u\|_b^p}{\int\limits_\Gamma g(x,u)u\,d\Gamma},$$

where $\|\cdot\|_b$ is defined on E as follows

$$\|u\|_b := \left(\int\limits_\Omega a(x)|Du|^p\,dx + \int\limits_\Gamma b(x)|u|^p\,d\Gamma\right)^{1/p}. \tag{3}$$

We introduce the convention that if $\mathcal{N}_g = \emptyset$ then $\theta_* = -\infty$ and $\theta^* = +\infty$, provided $\mathcal{P}_g = \emptyset$. Define

$$\mathcal{N}_G := \{u \in X : \int\limits_\Gamma G(x,u)\,d\Gamma < 0\}, \qquad \mathcal{P}_G := \{u \in X : \int\limits_\Gamma G(x,u)\,d\Gamma > 0\}$$

$$\theta_- := \sup_{u\in\mathcal{N}_G} \frac{\|u\|_b^p}{p\int\limits_\Gamma G(x,u)\,d\Gamma}, \qquad \theta^+ := \inf_{u\in\mathcal{P}_G} \frac{\|u\|_b^p}{p\int\limits_\Gamma G(x,u)\,d\Gamma}.$$

If $\mathcal{N}_G = \emptyset$ (resp., $\mathcal{P}_G = \emptyset$) then we set $\theta_- = -\infty$ (resp., $\theta^+ = +\infty$).

Our main results are the following

Theorem 1 *Suppose $\theta_* < \theta < \theta^*$. Then there exists $\lambda^* > 0$ such that problem $(P_{\lambda,\theta})$ has no solution, provided that $0 < \lambda < \lambda^*$.*

In order to state the next result, define $\underline{\theta} = \max\{\theta_*, \theta_-\}$ if $g(x,\cdot)$ is odd and $\underline{\theta} = 0$ elsewhere. Let $\overline{\theta} = \min\{\theta^*, \theta^+\}$ and observe that $\underline{\theta} \le 0 \le \overline{\theta}$. Set $J = (\underline{\theta}, \overline{\theta})$ and assume that $J \ne \emptyset$.

Theorem 2 *Suppose $\theta \in J$. Then there exists $\lambda_0 > 0$ such that the following hold:*

(i) Problem $(P_{\lambda,\theta})$ admits a solution, for any $\lambda \ge \lambda_0$;

(ii) Problem $(P_{\lambda,\theta})$ does not have any solution, provided that $0 < \lambda < \lambda_0$.

3 Auxiliary results

We first prove that the energy functional $\Phi_{\lambda,\theta}$ is well defined on X.

Lemma 1 *There exist positive constants C_1 and C_2 such that for every $u \in E$*

$$\int\limits_\Omega |u|^p v_0(x)\,dx \le C_1 \int\limits_\Omega |Du|^p v_1(x)\,dx + C_2 \int\limits_\Gamma |n\cdot x||u|^p v_0(x)\,d\Gamma.$$

Proof. Using the divergence theorem we obtain, for any $u \in C_\delta^\infty(\Omega)$,

$$\int\limits_\Omega x\cdot D(|u|^p v_0(x))\,dx = \int\limits_\Gamma (n\cdot x)|u|^p v_0(x)\,d\Gamma - N\int\limits_\Omega |u|^p v_0(x)\,dx.$$

This implies

$$N\int\limits_\Omega |u|^p v_0(x)\,dx \le \int\limits_\Gamma |n\cdot x||u|^p v_0(x)\,d\Gamma + \int\limits_\Omega |u|^p|x||Dv_0(x)|\,dx + p\int\limits_\Omega |x||u|^{p-1}|Du|v_0(x)\,dx. \tag{4}$$

Using Hölder's and Young's inequality, we get the estimate

$$p\int_\Omega |x||u|^{p-1}|Du|v_0(x)\,dx \le \; p\left(\int_\Omega |u|^p v_0(x)\,dx\right)^{(p-1)/p}\left(\int_\Omega |Du|^p|x|^p v_0(x)\,dx\right)^{1/p} \le \varepsilon(p-1)\int_\Omega |u|^p v_0(x)\,dx + \varepsilon^{1-p}\int_\Omega |Du|^p|x|^p v_0\,dx \tag{5}$$

where $\varepsilon > 0$ is an arbitrary real number. From (4), (5) and $(\mathbf{H_1})$ it follows that

$$(N-\varepsilon(p-1)-\sigma)\int_\Omega |u|^p v_0(x)\,dx \le \varepsilon^{1-p}\int_\Omega |Du|^p|x|^p v_0(x)\,dx + \int_\Gamma |n\cdot x||u|^p v_0(x)\,d\Gamma.$$

Using $(\mathbf{H_2})$ and choosing ε small enough we find

$$\int_\Omega |u|^p v_0(x)\,dx \le C_1\int_\Omega |Du|^p v_1(x)\,dx + C_2\int_\Gamma |n\cdot x||u|^p v_0(x)\,d\Gamma, \qquad \forall u\in C_\delta^\infty(\Omega).$$

The conclusion of our lemma follows now by standard density arguments. □

Lemma 2 *The quantity $\|\cdot\|_b$ defined by (3) represents an equivalent norm on E.*

Proof. The inequality $\|u\|_E^p \le c\|u\|_b^p$ follows directly from Lemma 1 by using the left hand side inequalities which appear in hypotheses $(\mathbf{H_2})$ and $(\mathbf{H_3})$.

By Remark 2 we know that the trace $E\to L^p(\Gamma;\, w_0)$ is continuous. Therefore, we have that there exists $C>0$ such that

$$\int_\Gamma |u|^p w_0(x)\,d\Gamma \le C\|u\|_E^p \qquad \forall u\in E. \tag{6}$$

Using the inequalities remained in $(\mathbf{H_2})$, $(\mathbf{H_3})$ and by (6) it follows that

$$\|u\|_b^p \le a_1\int_\Omega |Du|^p v_1(x)\,dx + b_1\int_\Gamma |u|^p w_0(x)\,d\Gamma \le c'\|u\|_E^p.$$

Hence the desired equivalence is proved. □

For $\lambda>0$ fixed, let f_λ be the function defined by

$$f_\lambda(x,s) = f(\lambda,x,s) \quad \forall x\in\Omega,\ \forall s\in\mathbf{R}.$$

Set $F_\lambda(x,u) = \int_0^u f_\lambda(x,s)\,ds$. Denote by N_{f_λ}, N_{F_λ}, N_g, N_G the corresponding Nemytskii operators.

Lemma 3 *The operators*

$$N_{f_\lambda} : L^q(\Omega;\, w_1) \to L^{q/(q-1)}(\Omega;\, w_1^{1/(1-q)}), \qquad N_{F_\lambda} : L^q(\Omega;\, w_1)\to L^1(\Omega)$$
$$N_g : L^m(\Gamma;\, w_2) \to L^{m/(m-1)}(\Gamma;\, w_2^{1/(1-m)}), \qquad N_G : L^m(\Gamma;\, w_2)\to L^1(\Gamma)$$

are bounded and continuous.

Proof. From hypothesis $(\mathbf{H_5})$ we deduce that

$$\begin{aligned} |f_\lambda(x,u)| &\le \frac{\lambda}{q-1}\varphi(x)|u|^{q-1} \le \tilde{C}_f\lambda|u|^{q-1}w_1(x) && \text{a.e. } x\in\Omega,\ \forall u\in\mathbf{R} \\ |F_\lambda(x,u)| &\le \frac{\lambda}{q(q-1)}\varphi(x)|u|^{q} \le \frac{\tilde{C}_f}{q}\lambda|u|^{q}w_1(x) && \text{a.e. } x\in\Omega,\ \forall u\in\mathbf{R}, \end{aligned} \tag{7}$$

where $\tilde{C}_f$ denotes $c_f/(q-1)$.

For $u\in L^q(\Omega;\,w_1)$ we get (setting $q'=q/(q-1)$)

$$\int_\Omega |N_{f_\lambda}(u)|^{q'}w_1^{1/(1-q)}\,dx \le (\tilde{C}_f\lambda)^{q'}\int_\Omega |u|^q w_1(x)\,dx.$$

Therefore, N_{f_λ} is bounded. Similarly, the boundedness of N_{F_λ} follows from the estimate

$$\int_\Omega |N_{F_\lambda}(u)|\,dx \le \frac{\tilde{C}_f}{q}\lambda\int_\Omega |u|^q w_1(x)\,dx.$$

Let $m'=m/(m-1)$ and $u\in L^m(\Gamma;\,w_2)$. Then, by $(\mathbf{H_9})$

$$\begin{aligned} \int_\Gamma |N_g(u)|^{m'}w_2^{1/(1-m)}\,d\Gamma \le\ & 2^{m'-1}\left(\int_\Gamma g_0^{m'}w_2^{1/(1-m)}\,d\Gamma + \int_\Gamma g_1^{m'}|u|^m w_2^{1/(1-m)}\,d\Gamma\right) \le \\ & 2^{m'-1}\left(C + C_g^{m'}\int_\Gamma |u|^m w_2(x)\,d\Gamma\right), \end{aligned}$$

which shows that N_g is bounded. In a similar way, by $(\mathbf{H_9})$ and Hölder's inequality we obtain

$$\begin{aligned} \int_\Gamma |N_G(u)|\,d\Gamma \le\ & \int_\Gamma g_0|u|\,d\Gamma + \frac{1}{m}\int_\Gamma g_1|u|^m\,d\Gamma \le \\ & \left(\int_\Gamma g_0^{m'}w_2^{1/(1-m)}\,d\Gamma\right)^{1/m'}\cdot\left(\int_\Gamma |u|^m w_2(x)\,d\Gamma\right)^{1/m} + \frac{C_g}{m}\int_\Gamma |u|^m w_2(x)\,d\Gamma \end{aligned}$$

and the boundedness of N_G follows.

From the usual properties of Nemytskii operators we deduce the continuity of N_{f_λ}, N_{F_λ}, N_g and N_G (see Vainberg [21]). □

In view of Lemmas 2 and 3, $\Phi_{\lambda,\theta}$ is well defined on X.

Lemma 4 *The functional $\Phi_{\lambda,\theta}$ is Fréchet-differentiable on X.*

Proof. We use the notation

$$I(u)=\frac{1}{p}\|u\|_b^p,\quad J(u)=\frac{1}{r}\|u\|_{r,\Omega,h}^r,\quad K_G(u)=\int_\Gamma G(x,u)\,d\Gamma,\quad K_{F_\lambda}(u)=\int_\Omega F_\lambda(x,u)\,dx.$$

Then the Gâteaux derivative of $\Phi_{\lambda,\theta}$ is given by

$$\langle \Phi'_{\lambda,\theta}(u), v\rangle = \langle I'(u), v\rangle + \langle J'(u), v\rangle - \langle K'_{F_\lambda}(u), v\rangle - \theta \langle K'_G(u), v\rangle\,,$$

where

$$\langle I'(u), v\rangle = \int_\Omega a(x)|Du|^{p-2}Du\cdot Dv\,dx + \int_\Gamma b(x)|u|^{p-2}uv\,d\Gamma,$$

$$\langle J'(u), v\rangle = \int_\Omega h(x)|u|^{r-2}uv\,dx, \quad \langle K'_{F_\lambda}(u), v\rangle = \int_\Omega f_\lambda(x,u)v\,dx, \quad \langle K'_G(u), v\rangle = \int_\Gamma g(x,u)v\,d\Gamma.$$

We need only to show the continuity of $\Phi'_{\lambda,\theta}$ and the assertion is proved.

Clearly, $I' : E \to E'$ and $J' : L^r(\Omega;\, h) \to (L^r(\Omega;\, h))'$ are continuous. By using (2) we see immediately that $I' : X \to X'$ and $J' : X \to X'$ are continuous.

The operator K'_G is a composition of operators

$$X \overset{i}{\hookrightarrow} E \xrightarrow{\gamma} L^m(\Gamma;\, w_2) \xrightarrow{N_g} L^{m/(m-1)}(\Gamma;\, w_2^{1/(1-m)}) \xrightarrow{k} E' \overset{i'}{\hookrightarrow} X'$$

where $\langle k(u), v\rangle = \int_\Gamma uv\,d\Gamma$. Obviously, k is a linear operator. By Hölder's inequality and Remark 2,

$$\int_\Gamma |uv|\,d\Gamma \le \left(\int_\Gamma |u|^{m'} w_2^{1/(1-m)}\,d\Gamma\right)^{1/m'} \cdot \left(\int_\Gamma |v|^m w_2\,d\Gamma\right)^{1/m} \le C\|u\|_{m/(m-1),\Gamma,w_2^{1/(1-m)}}\|v\|_E\,,$$

which shows that k is continuous. As a composition of continuous operators, K'_G is continuous, too. Moreover, it is compact since the trace operator γ is compact. In a similar way we obtain that K'_{F_λ} is continuous such that the Fréchet-differentiability of $\Phi_{\lambda,\theta}$ follows. □

4 Proof of Theorem 1

Assume $\theta_* < \theta < \theta^*$ and let $\lambda > 0$ be chosen such that problem $(P_{\lambda,\theta})$ possesses at least a solution. We claim that there exists $\lambda^* > 0$ such that $\lambda \ge \lambda^*$. Suppose that u is a solution of problem $(P_{\lambda,\theta})$. Then, using (7) we find

$$\|u\|_b^p - \theta\int_\Gamma g(x,u)u\,d\Gamma + \int_\Omega h(x)|u|^r\,dx = \int_\Omega f(\lambda,x,u)u\,dx \le \lambda\tilde{C}_f\int_\Omega |u|^q w_1(x)\,dx\,. \tag{8}$$

Now, the Young inequality implies the following estimate

$$\lambda\tilde{C}_f\int_\Omega |u|^q w_1(x)\,dx = \int_\Omega \frac{\lambda\tilde{C}_f w_1}{h^{q/r}}\cdot h^{q/r}|u|^q\,dx \le \frac{r-q}{r}(\tilde{C}_f\lambda)^{r/(r-q)}\int_\Omega\left(\frac{w_1^r}{h^q}\right)^{1/(r-q)}dx + \frac{q}{r}\int_\Omega h|u|^r\,dx\,.$$

This inequality combined with (8) gives

$$\begin{aligned}\|u\|_b^p - \theta\int_\Gamma g(x,u)u\,d\Gamma \le\ & \frac{r-q}{r}(\tilde{C}_f\lambda)^{r/(r-q)}\int_\Omega\left(\frac{w_1^r}{h^q}\right)^{1/(r-q)}dx + \frac{q-r}{r}\int_\Omega h|u|^r\,dx \le \\ & \frac{r-q}{r}(\tilde{C}_f\lambda)^{r/(r-q)}\int_\Omega\left(\frac{w_1^r}{h^q}\right)^{1/(r-q)}dx\,.\end{aligned} \tag{9}$$

On the hand, $\theta < \theta^*$ implies the existence of a constant $C_1 \in (0,1)$ such that

$$\theta \leq (1-C_1)\theta^* \leq (1-C_1)\frac{\|u\|_b^p}{\int\limits_\Gamma g(x,u)u\,d\Gamma} \qquad \text{for all } u \in \mathcal{P}_g$$

which yields

$$\|u\|_b^p - \theta \int\limits_\Gamma g(x,u)u\,d\Gamma \geq C_1\|u\|_b^p \qquad \text{for all } u \in \mathcal{P}_g. \tag{10}$$

On the other hand, $\theta_* < \theta$ shows that there exists $C_2 \in (0,1)$ such that

$$\|u\|_b^p - \theta \int\limits_\Gamma g(x,u)u\,d\Gamma \geq C_2\|u\|_b^p \qquad \text{for all } u \in \mathcal{N}_g. \tag{11}$$

From (10) and (11) we conclude that

$$\|u\|_b^p - \theta \int\limits_\Gamma g(x,u)u\,d\Gamma \geq C_0\|u\|_b^p \qquad \text{for all } u \in X \tag{12}$$

where $C_0 = \min\{C_1, C_2\}$.

The continuity of the embedding $E \hookrightarrow L^q(\Omega;\, w_1)$ implies the existence of $\overline{C} > 0$ such that

$$\overline{C}\|u\|_{q,\Omega,w_1}^p \leq \|u\|_b^p \qquad \text{for all } u \in E.$$

By (8) and (12) we have

$$C_0\overline{C}\left(\int\limits_\Omega |u|^q w_1(x)\,dx\right)^{p/q} \leq C_0\|u\|_b^p \leq \lambda\tilde{C}_f \int\limits_\Omega |u|^q w_1(x)\,dx, \tag{13}$$

which implies

$$(\overline{C}C_0\tilde{C}_f^{-1}\lambda^{-1})^{q/(q-p)} \leq \int\limits_\Omega |u|^q w_1(x)\,dx.$$

This combined with (13) yields

$$C_0\overline{C}(\overline{C}C_0\tilde{C}_f^{-1}\lambda^{-1})^{p/(q-p)} \leq C_0\|u\|_b^p. \tag{14}$$

Using (14) together with (9) and (12) we obtain

$$C_0\overline{C}(\overline{C}C_0\tilde{C}_f^{-1}\lambda^{-1})^{p/(q-p)} \leq \frac{r-q}{r}(\tilde{C}_f\lambda)^{r/(r-q)} \int\limits_\Omega \left(\frac{w_1^r}{h^q}\right)^{1/(r-q)} dx\,.$$

We see that our claim follows if we take

$$\lambda^* = C^*\left(\int\limits_\Omega \left(\frac{w_1^r}{h^q}\right)^{1/(r-q)} dx\right)^{-(q-p)(r-q)/q(r-p)}$$

where C^* denotes $\tilde{C}_f^{-1}\left[C_0\overline{C}\left(\frac{r}{r-q}\right)^{(q-p)/q}\right]^{(r-q)/(r-p)}$. □

Corollary 1 *Suppose $\theta_* < \theta < \theta^*$ and $\lambda > 0$ such that $(P_{\lambda,\theta})$ has a solution u. Then*

$$C_0\|u\|_b^p + \frac{r-q}{r}\int_\Omega h|u|^r\,dx \le \frac{r-q}{r}(\tilde{C}_f\lambda)^{r/(r-q)}\int_\Omega \left(\frac{w_1^r}{h^q}\right)^{1/(r-q)}dx$$

and

$$\|u\|_b \ge K\lambda^{-1/(q-p)},$$

where $K > 0$ is a constant independent of u.

Proof. The first part of the assertion follows by (9) and (12). The second one is implied by (14) which shows that the constant K can be chosen, for example as $\overline{C}^{q/p(q-p)}(C_0\tilde{C}_f^{-1})^{1/(q-p)}$. □

5 Properties of $\Phi_{\lambda,\theta}$

Proceeding in the same manner as we did for proving (12) we can show that if we take $\theta_- < \theta < \theta^+$ then there exists $c > 0$ such that

$$\frac{1}{p}\|u\|_b^p - \theta\int_\Gamma G(x,u)\,d\Gamma \ge c\|u\|_b^p \qquad \text{for all } u \in X. \tag{15}$$

We shall employ in what follows the following elementary inequality

$$s|u|^\mu - t|u|^\nu \le C_{\mu,\nu}s\left(\frac{s}{t}\right)^{\mu/(\nu-\mu)} \qquad \forall u \in \mathbf{R},\ \forall s,t \in (0,\infty),\ \forall 0 < \mu < \nu\,. \tag{16}$$

Lemma 5 *Suppose $\theta_- < \theta < \theta^+$ and $\lambda > 0$ is arbitrary. Then the functional $\Phi_{\lambda,\theta}$ is coercive.*

Proof. From (7) we have that there exists $C > 0$ such that

$$F(\lambda,x,u) \le C\lambda|u|^q w_1(x) \qquad \text{a.e. } x \in \Omega,\ \forall u \in \mathbf{R}. \tag{17}$$

By virtue of (16) and $(\mathbf{H_{10}})$ we obtain

$$\int_\Omega \left(C\lambda w_1|u|^q - \frac{h}{2r}|u|^r\right)dx \le C_{r,q}\int_\Omega \lambda w_1\left(\frac{\lambda w_1}{h}\right)^{q/(r-q)}dx = C_{r,q}\lambda^{r/(r-q)}\int_\Omega\left(\frac{w_1^r}{h^q}\right)^{1/(r-q)}dx \le C'.$$

Using (15), (17) and the above estimate we find

$$\begin{aligned}\Phi_{\lambda,\theta}(u) = {} & \frac{1}{p}\|u\|_b^p - \theta\int_\Gamma G(x,u)\,d\Gamma - \int_\Omega F(\lambda,x,u)\,dx + \frac{1}{r}\int_\Omega h|u|^r\,dx \ge \\ & c\|u\|_b^p - \int_\Omega\left(C\lambda|u|^q w_1 - \frac{h}{2r}|u|^r\right)dx + \frac{1}{2r}\int_\Omega h|u|^r\,dx \ge c\|u\|_b^p + \frac{1}{2r}\int_\Omega h|u|^r\,dx - C'\end{aligned}$$

and the coercivity of $\Phi_{\lambda,\theta}$ follows. □

Lemma 6 *Suppose $\theta_- < \theta < \theta^+$, $\lambda > 0$ is arbitrary and $\{u_n\}$ is a sequence in X such that $\Phi_{\lambda,\theta}(u_n)$ is bounded. Then there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that*

$$u_n \rightharpoonup u_0 \textit{ in } X, \quad u_n \to u_0 \textit{ a.e. in } \Omega \quad \textit{and} \quad \Phi_{\lambda,\theta}(u_0) \le \liminf_{n\to\infty} \Phi_{\lambda,\theta}(u_n).$$

Proof. In view of Lemma 5, the boundedness of $\Phi_{\lambda,\theta}(u_n)$ shows that $\{u_n\}$ must be bounded in X. Using (2) and Remark 2 we may assume (up to a subsequence) that

$$u_n \rightharpoonup u_0 \text{ in } X, \quad u_n \to u_0 \text{ in } L^q(\Omega;\, w_1) \quad \text{and} \quad u_n \to u_0 \text{ a.e. in } \Omega.$$

Set

$$\Xi(x,u) = F(\lambda,x,u) - \frac{1}{r}h|u|^r \quad \text{and} \quad \xi(x,u) = \Xi_u(x,u).$$

By hypothesis $(\mathbf{H_5})$ and (16) we obtain

$$\xi_u(x,u) = f_u(\lambda,x,u) - (r-1)h|u|^{r-2} \le \lambda c_f w_1 |u|^{q-2} - (r-1)h|u|^{r-2} \le C\lambda w_1 \left(\frac{\lambda w_1}{h}\right)^{(q-2)/(r-q)}.$$

It follows that

$$\int_\Omega (\Xi(x,u_n) - \Xi(x,u_0))\,dx = \int_\Omega \left(\int_0^1 \int_0^s \xi_u(x, u_0 + t(u_n-u_0))\,dt\,ds\right)(u_n-u_0)^2\,dx \le C' \int_\Omega \frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}(u_n-u_0)^2\,dx.$$

This inequality will be used to get the estimate for $\Phi_{\lambda,\theta}(u_0) - \Phi_{\lambda,\theta}(u_n)$:

$$\begin{aligned}
&\Phi_{\lambda,\theta}(u_0) - \Phi_{\lambda,\theta}(u_n) = \\
&\frac{1}{p}(\|u_0\|_b^p - \|u_n\|_b^p) + \theta \int_\Gamma (G(x,u_n) - G(x,u_0))\,d\Gamma + \int_\Omega (\Xi(x,u_n) - \Xi(x,u_0))\,dx \le \\
&\frac{1}{p}(\|u_0\|_b^p - \|u_n\|_b^p) + \theta \int_\Gamma (G(x,u_n) - G(x,u_0))\,d\Gamma + C' \int_\Omega \frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}(u_n-u_0)^2\,dx.
\end{aligned}$$

The compactness of the trace operator $E \to L^m(\Gamma;\, w_2)$ and the continuity of the Nemytskii operator $N_G : L^m(\Gamma;\, w_2) \to L^1(\Gamma)$ imply that $N_G(u_n) \to N_G(u_0)$ in $L^1(\Gamma)$ i.e. $\int_\Gamma |N_G(u_n) - N_G(u_0)|\,d\Gamma \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \int_\Gamma G(x,u_n)\,d\Gamma = \int_\Gamma G(x,u_0)\,d\Gamma. \tag{18}$$

By Hölder's inequality we find

$$\int_\Omega \frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}(u_n-u_0)^2\,dx \le \left(\int_\Omega \left(\frac{w_1^r}{h^q}\right)^{1/(r-q)} dx\right)^{(q-2)/q} \cdot \left(\int_\Omega |u_n-u_0|^q w_1(x)\,dx\right)^{2/q}.$$

Since $u_n \to u_0$ in $L^q(\Omega;\, w_1)$ we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}(u_n-u_0)^2\,dx = 0. \tag{19}$$

The lower semicontinuity of $\|\cdot\|_b$ with respect to the weak topology, (18) and (19) finish the proof. □

Lemma 7 *Suppose $\theta_* < \theta < \theta^*$ and $\lambda_n \searrow \lambda_0 > 0$ such that problem $(P_{\lambda_n,\theta})$ has a solution u_n for each n. Then $\{u_n\}$ converges weakly (up to a subsequence) in X to some u_0 which is a non-negative critical point of $\Phi_{\lambda_0,\theta}$.*

Proof. By Corollary 1, $\{u_n\}$ is bounded in X. Therefore, in view of Remark 2, Lemma 2 and (2), we may assume (passing eventually to subsequences) that

$$u_n \rightharpoonup u_0 \text{ in } X,\ u_n \rightharpoonup u_0 \text{ in } L^r(\Omega;\, h),\ u_n \rightharpoonup u_0 \text{ in } E,\ u_n \rightharpoonup u_0 \text{ in } L^p(\Gamma;\, b),\ \frac{\partial u_n}{\partial x_i} \rightharpoonup \frac{\partial u_0}{\partial x_i} \text{ in } L^p(\Omega;\, a) \quad (20)$$

$$u_n \to u_0 \text{ in } L^q(\Omega;\, w_1), \quad u_n \to u_0 \text{ in } L^m(\Gamma;\, w_2), \quad u_n \to u_0 \text{ a.e. in } \Omega, \quad u_n \to u_0 \text{ a.e. in } \Gamma\,. \quad (21)$$

We now observe that the embedding $E \hookrightarrow L^s_{\mathrm{loc}}(\Omega)$ is compact for all $p \le s < p^*$. This and (20) imply

$$u_n \to u_0 \text{ in } L^s_{\mathrm{loc}}(\Omega), \quad \forall p \le s < p^*. \quad (22)$$

Since u_n is a non-negative critical point of $\Phi_{\lambda_n,\theta}$ for each n, we derive by (21) that $u_0 \ge 0$ in Ω and for any $v \in X$ we have

$$\int_\Omega a|Du_n|^{p-2}Du_n\cdot Dv\,dx + \int_\Gamma b|u_n|^{p-2}u_n v\,d\Gamma + \int_\Omega h|u_n|^{r-2}u_n v\,dx = \int_\Omega f(\lambda_n,x,u_n)v\,dx + \theta\int_\Gamma g(x,u_n)v\,d\Gamma\,.$$

By (20) we find that $\{|u_n|^{r-2}u_n\}$ is bounded in $L^{r/(r-1)}(\Omega;\, h)$, while by (21) we have that $|u_n|^{r-2}u_n \to |u_0|^{r-2}u_0$ a.e. in Ω. Combining these facts we get

$$|u_n|^{r-2}u_n \rightharpoonup |u_0|^{r-2}u_0 \text{ in } L^{r/(r-1)}(\Omega;\, h). \quad (23)$$

For $v \in L^r(\Omega;\, h)$ fixed, set $l_v(u) = \int_\Omega huv\,dx$, for all $u \in L^{r/(r-1)}(\Omega;\, h)$. It is easy to verify that $l_v \in (L^{r/(r-1)}(\Omega;\, h))'$. This together with (23) implies

$$\lim_{n\to\infty}\int_\Omega h|u_n|^{r-2}u_n v\,dx = \int_\Omega h|u_0|^{r-2}u_0 v\,dx, \qquad \forall v \in X. \quad (24)$$

Similarly

$$\lim_{n\to\infty}\int_\Gamma b|u_n|^{p-2}u_n v\,d\Gamma = \int_\Gamma b|u_0|^{p-2}u_0 v\,d\Gamma, \qquad \forall v \in X. \quad (25)$$

Taking into account (21) and Lemma 3 we infer that

$$N_{f_{\lambda_0}}(u_n) \to N_{f_{\lambda_0}}(u_0) \text{ in } L^{q/(q-1)}(\Omega;\, w_1^{1/(1-q)}) \text{ and } N_g(u_n) \to N_g(u_0) \text{ in } L^{m/(m-1)}(\Gamma;\, w_2^{1/(1-m)}). \quad (26)$$

By Hölder's inequality and $(\mathbf{H_7})$ we derive the estimates

$$\begin{aligned}
&\int_\Omega |(f(\lambda_n,x,u_n) - f(\lambda_0,x,u_0))v|\,dx \le \\
&\int_\Omega |(f(\lambda_n,x,u_n) - f(\lambda_0,x,u_n))v|\,dx + \int_\Omega |(f(\lambda_0,x,u_n) - f(\lambda_0,x,u_0))v|\,dx \le \\
&C_f|\lambda_n - \lambda_0|\int_\Omega |u_n|^{q-1}|v|w_1\,dx + \int_\Omega |(N_{f_{\lambda_0}}(u_n) - N_{f_{\lambda_0}}(u_0))v|\,dx \le \\
&C_f|\lambda_n - \lambda_0|\|u_n\|^{q-1}_{q,\Omega,w_1}\|v\|_{q,\Omega,w_1} + \|N_{f_{\lambda_0}}(u_n) - N_{f_{\lambda_0}}(u_0)\|_{q/(q-1),\Omega,w_1^{1/(1-q)}}\|v\|_{q,\Omega,w_1}
\end{aligned}$$

and

$$\int_\Gamma |(g(x,u_n)-g(x,u_0))v|\,d\Gamma \le \|N_g(u_n)-N_g(u_0)\|_{m/(m-1),\Gamma,w_2^{1/(1-m)}}\|v\|_{m,\Gamma,w_2}.$$

Then, in virtue of (26) we find

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega f(\lambda_n,x,u_n)v\,dx = \int_\Omega f(\lambda_0,x,u_0)v\,dx\,, \qquad \forall v\in X.\\ &\lim_{n\to\infty}\int_\Gamma g(x,u_n)v\,d\Gamma = \int_\Gamma g(x,u_0)v\,d\Gamma\,, \qquad \forall v\in X. \end{aligned} \tag{27}$$

We now claim that $Du_n \to Du_0$ a.e. in Ω. Set

$$\Omega_R = \{x\in \mathbf{R}^N:\ |x|<R \text{ and } \operatorname{dist}(x,\mathbf{R}^N\setminus\Omega) > \frac{1}{R}\}.$$

It is clear that there exists $R_0>0$ such that $\Omega_R\neq\emptyset$ for all $R>R_0$. Since $\Omega_R\subset\Omega_{R'}\subset\subset\Omega$ for all $R_0\le R<R'$ and $\cup_{R\ge R_0}\Omega_R=\Omega$ we need only to show

$$Du_n\to Du_0 \quad \text{a.e. in } \Omega_R \text{ for any } R\ge R_0.$$

For this purpose we use the following inequalities (see Diaz [8, Lemma 4.10]) that hold for any $\xi,\zeta\in\mathbf{R}^N$

$$|\xi-\zeta|^p \le C(|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta)(\xi-\zeta), \qquad \text{for } p\ge 2\,; \tag{28}$$

$$|\xi-\zeta|^2 \le C(|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta)(\xi-\zeta)(|\xi|+|\zeta|)^{2-p}, \qquad \text{for } 1<p<2\,. \tag{29}$$

Therefore, it is sufficient to prove that

$$(|Du_n|^{p-2}Du_n-|Du_0|^{p-2}Du_0)\cdot(Du_n-Du_0)\to 0 \quad \text{a.e. in } \Omega_R \text{ for any } R\ge R_0. \tag{30}$$

For a fixed $R\ge R_0$, choose $\vartheta\in C_0^\infty(\mathbf{R}^N)$ with $0\le\vartheta\le 1$ in $\mathbf{R}^N$, $\vartheta\equiv 1$ on Ω_R and $\vartheta\equiv 0$ on $\mathbf{R}^N\setminus\Omega_{2R}$. Then by (20) and (21) we have that $\vartheta u_n \rightharpoonup \vartheta u_0$ in E which yields

$$\int_\Omega a|Du_0|^{p-2}Du_0\cdot D(\vartheta u_n-\vartheta u_0)\,dx + \int_\Gamma b\vartheta|u_0|^{p-2}u_0(u_n-u_0)\,d\Gamma \to 0. \tag{31}$$

By Hölder's inequality and (22) we find

$$|\int_\Omega a(u_n-u_0)|Du_0|^{p-2}Du_0\cdot D\vartheta\,dx| \le C_1\left(\int_{\operatorname{Supp}\vartheta} a|Du_0|^p\,dx\right)^{(p-1)/p}\left(\int_{\operatorname{Supp}\vartheta}|u_n-u_0|^p\,dx\right)^{1/p} \to 0.$$

Using this fact in (31) we obtain

$$\int_\Omega a\vartheta|Du_0|^{p-2}Du_0\cdot D(u_n-u_0)\,dx + \int_\Gamma b\vartheta|u_0|^{p-2}u_0(u_n-u_0)\,d\Gamma \to 0. \tag{32}$$

On the other hand, since $\langle \Phi'_{\lambda_n,\theta}(u_n),\vartheta(u_n-u_0)\rangle = 0$ we have

$$\int_\Omega a\vartheta|Du_n|^{p-2}Du_n\cdot D(u_n-u_0)\,dx + \int_\Gamma b\vartheta|u_n|^{p-2}u_n(u_n-u_0)\,d\Gamma + \int_\Omega a(u_n-u_0)|Du_n|^{p-2}Du_n\cdot D\vartheta\,dx =$$

$$\int_\Omega h\vartheta|u_n|^{r-2}u_n(u_0-u_n)\,dx+\int_\Omega f(\lambda_n,x,u_n)\vartheta(u_n-u_0)\,dx+\theta\int_\Gamma g(x,u_n)\vartheta(u_n-u_0)\,d\Gamma.$$

By Hölder's inequality, (20) and (22) we derive

$$\left|\int_\Omega a(u_n-u_0)|Du_n|^{p-2}Du_n\cdot D\vartheta\,dx\right|\le C_1\left(\int_{\mathrm{Supp}\,\vartheta} a|Du_n|^p\,dx\right)^{(p-1)/p}\left(\int_{\mathrm{Supp}\,\vartheta}|u_n-u_0|^p\,dx\right)^{1/p}\to 0$$

and

$$\left|\int_\Omega h\vartheta|u_n|^{r-2}u_n(u_0-u_n)\,dx\right|\le C_2\left(\int_{\mathrm{Supp}\,\vartheta} h|u_n|^r\,dx\right)^{(r-1)/r}\left(\int_{\mathrm{Supp}\,\vartheta}|u_n-u_0|^r\,dx\right)^{1/r}\to 0.$$

By (7), (21), (26) and Hölder's inequality we see that

$$\begin{aligned}\left|\int_\Omega f(\lambda_n,x,u_n)\vartheta(u_n-u_0)\,dx\right|\le\ & \tilde{C}_f\sup_{n\ge1}\lambda_n\int_\Omega|u_n|^{q-1}|u_n-u_0|w_1\,dx\le\\ & \tilde{C}_f\sup_{n\ge1}\lambda_n\|u_n\|_{q,\Omega,w_1}^{q-1}\|u_n-u_0\|_{q,\Omega,w_1}\to 0\end{aligned}$$

and

$$\left|\int_\Gamma g(x,u_n)\vartheta(u_n-u_0)\,d\Gamma\right|\le\|N_g(u_n)\|_{m/(m-1),\Gamma,w_2^{1/(1-m)}}\|u_n-u_0\|_{m,\Gamma,w_2}\to 0.$$

It follows that

$$\int_\Omega a\vartheta|Du_n|^{p-2}Du_n\cdot D(u_n-u_0)\,dx+\int_\Gamma b\vartheta|u_n|^{p-2}u_n(u_n-u_0)\,d\Gamma\to 0. \tag{33}$$

Since

$$\begin{aligned}&0\le\int_\Omega a\vartheta(|Du_n|^{p-2}Du_n-|Du_0|^{p-2}Du_0)\cdot(Du_n-Du_0)\,dx\le\\ &\int_\Omega a\vartheta(|Du_n|^{p-2}Du_n-|Du_0|^{p-2}Du_0)\cdot(Du_n-Du_0)\,dx+\int_\Gamma b\vartheta(|u_n|^{p-2}u_n-|u_0|^{p-2}u_0)(u_n-u_0)\,d\Gamma\end{aligned}$$

we deduce by (32) and (33) that

$$\lim_{n\to\infty}\int_{\Omega_R} a(|Du_n|^{p-2}Du_n-|Du_0|^{p-2}Du_0)\cdot(Du_n-Du_0)\,dx=0.$$

Hence (30) holds. Therefore, the claim that $Du_n\to Du_0$ a.e. in Ω is proved. This combined with the fact that $\{|Du_n|^{p-2}\frac{\partial u_n}{\partial x_i}\}$ is bounded in $L^{p/(p-1)}(\Omega;\,a)$ implies

$$|Du_n|^{p-2}\frac{\partial u_n}{\partial x_i}\rightharpoonup|Du_0|^{p-2}\frac{\partial u_0}{\partial x_i}\quad\text{in } L^{p/(p-1)}(\Omega;\,a).$$

It follows that

$$\lim_{n\to\infty}\int_\Omega a|Du_n|^{p-2}Du_n\cdot Dv\,dx=\int_\Omega a|Du_0|^{p-2}Du_0\cdot Dv\,dx,\qquad\forall v\in X. \tag{34}$$

By (24), (25), (27) and (34) we conclude that u_0 is a critical point of $\Phi_{\lambda_0,\theta}$. □

6 Proof of Theorem 2

Let $\theta \in J$ and $\lambda > 0$ be arbitrary. From Lemma 5 we see that $m_{\lambda,\theta} := \inf_{u \in X} \Phi_{\lambda,\theta}(u)$ is real. Let $\{u_n\}$ be a sequence such that $\lim_{n\to\infty} \Phi_{\lambda,\theta}(u_n) = m_{\lambda,\theta}$. According to Lemma 6, we can assume (up to a subsequence) that

$$u_n \rightharpoonup u_0 \text{ in } X \quad \text{and} \quad \Phi_{\lambda,\theta}(u_0) \leq \liminf_{n\to\infty} \Phi_{\lambda,\theta}(u_n) = m_{\lambda,\theta}.$$

This shows that $\inf_{u \in X} \Phi_{\lambda,\theta}(u)$ is attained in u_0. From $(\mathbf{H_4})$ and $(\mathbf{H_8})$ we deduce that $G(x, |u_0|) \geq G(x, u_0)$ a.e. $x \in \Gamma$ and $F(\lambda, x, |u_0|) \geq F(\lambda, x, u_0)$ a.e. $x \in \Omega$. It follows that $\Phi_{\lambda,\theta}(|u_0|) \leq \Phi_{\lambda,\theta}(u_0)$. Therefore, we may assume that $u_0 \geq 0$ on Ω. To ensure that $u_0 \not\equiv 0$ we shall prove that $m_{\lambda,\theta}$ is negative provided that $\lambda > \tilde{\lambda}$ for some $\tilde{\lambda} > 0$.

By hypothesis $(\mathbf{H_6})$ we deduce that there exists $\delta > 0$ independent of x and λ such that

$$F(\lambda, x, u(x)) \geq \frac{\lambda}{2q}|u(x)|^q w_1(x) \quad \text{a.e. } x \in \Omega,\ \forall u \in X \text{ with } \sup_{x\in\Omega} |u(x)| \leq \delta. \tag{35}$$

Set $\zeta > 0$ with the property that

$$Y = \{u \in X \setminus \{0\} : \ \sup_{x\in\Omega} |u(x)| \leq \zeta \|u\|_{q,\Omega,w_1}\} \neq \emptyset$$

and denote $\eta = \left(\frac{\delta}{\zeta}\right)^q$. Define

$$\tilde{\lambda} := \inf\{\frac{2q}{\eta p}\|u\|_b^p - \frac{2q}{\eta}\theta \int_\Gamma G(x,u)\,d\Gamma + \frac{2q}{\eta r}\int_\Omega h|u|^r\,dx : \ u \in Z\},$$

where

$$Z = \{u \in X : \ \sup_{x\in\Omega} |u(x)| \leq \delta, \ \int_\Omega |u|^q w_1(x)\,dx = \eta\}.$$

It is easy to verify that $Z \neq \emptyset$. Indeed, if $y \in Y$ then $u = \dfrac{\eta^{1/q}}{\|y\|_{q,\Omega,w_1}}\,y \in Z$.

We now claim that $\tilde{\lambda} > 0$. For this aim, we consider the constrained minimization problem

$$M := \inf\{\|u\|_b^p : \ u \in E, \int_\Omega |u|^q w_1(x)\,dx = \eta\}.$$

Since the embedding $E \hookrightarrow L^q(\Omega; w_1)$ is continuous, it follows that $M > 0$. Thus

$$\|u\|_b^p \geq M \qquad \text{for all } u \in X \text{ with } \int_\Omega |u|^q w_1(x)\,dx = \eta.$$

By applying the Hölder inequality we find

$$\int_\Omega |u|^q w_1\,dx = \int_\Omega \frac{w_1}{h^{q/r}} h^{q/r}|u|^q\,dx \leq \left(\int_\Omega \left(\frac{w_1^r}{h^q}\right)^{1/(r-q)} dx\right)^{(r-q)/r} \cdot \left(\int_\Omega h|u|^r\,dx\right)^{q/r}. \tag{36}$$

By virtue of (15) and (36) we have

$$\begin{aligned}\frac{2q}{\eta p}\|u\|_b^p - \frac{2q}{\eta}\theta\int_\Gamma G(x,u)\,d\Gamma + \frac{2q}{\eta r}\int_\Omega h|u|^r\,dx \geq{}& \frac{2q}{\eta}c\|u\|_b^p + \frac{2q}{\eta r}\int_\Omega h|u|^r\,dx \geq \\ & \frac{2q}{\eta}cM + \frac{2q}{\eta r}\eta^{r/q}\left(\int_\Omega \left(\frac{w_1^r}{h^q}\right)^{1/(r-q)} dx\right)^{-(r-q)/q}\end{aligned}$$

for all $u \in X$ with $\int_\Omega |u|^q w_1\,dx = \eta$. It follows that

$$\tilde{\lambda} \geq \frac{2q}{\eta}cM + \frac{2q}{r}\eta^{(r-q)/q}\left(\int_\Omega \left(\frac{w_1^r}{h^q}\right)^{1/(r-q)} dx\right)^{-(r-q)/q}$$

and our claim follows.

Let $\lambda > \tilde{\lambda}$. Then there exists a function $u_1 \in Z$ such that

$$\lambda > \frac{2q}{\eta p}\|u_1\|_b^p - \frac{2q}{\eta}\theta\int_\Gamma G(x,u_1)\,d\Gamma + \frac{2q}{\eta r}\int_\Omega h|u_1|^r\,dx.$$

This inequality and (35) imply

$$\begin{aligned}\Phi_{\lambda,\theta}(u_1) ={}& \frac{1}{p}\|u_1\|_b^p - \theta\int_\Gamma G(x,u_1)\,d\Gamma + \frac{1}{r}\int_\Omega h|u_1|^r\,dx - \int_\Omega F(\lambda,x,u_1(x))\,dx \leq \\ & \frac{1}{p}\|u_1\|_b^p - \theta\int_\Gamma G(x,u_1)\,d\Gamma + \frac{1}{r}\int_\Omega h|u_1|^r\,dx - \frac{\lambda}{2q}\int_\Omega |u_1|^q w_1\,dx < 0.\end{aligned}$$

Consequently, $\inf_{u\in X} \Phi_{\lambda,\theta}(u) < 0$. Thus, the problem $(P_{\lambda,\theta})$ has a solution if $\theta \in J$ and $\lambda > \tilde{\lambda}$.
Set

$$\lambda_0 = \inf\{\lambda > 0 : (P_{\lambda,\theta}) \text{ admits a solution}\}.$$

By Theorem 1, we see that $\lambda_0 \geq \lambda^* > 0$.

We now show that for each $\lambda > \lambda_0$ problem $(P_{\lambda,\theta})$ admits a solution. Indeed, for every $\lambda > \lambda_0$ there exists $\rho \in (\lambda_0, \lambda)$ such that problem $(P_{\rho,\theta})$ has a solution u_ρ which is a subsolution of problem $(P_{\lambda,\theta})$. We consider the variational problem

$$\inf\{\Phi_{\lambda,\theta}(u) : u \in X \text{ and } u \geq u_\rho\}.$$

By Lemmas 5 and 6 this problem admits a solution $\bar{u}$. This minimizer $\bar{u}$ is a solution of problem $(P_{\lambda,\theta})$. It remains to show that problem $(P_{\lambda_0,\theta})$ has also a solution. Let $\lambda_n \to \lambda_0$ and $\lambda_n > \lambda_0$ for each n. Problem $(P_{\lambda_n,\theta})$ has a solution u_n for each n. Then, in virtue of Lemma 7, we may assume (up to a subsequence) that $u_n \rightharpoonup u_0$ in X, $u_n \to u_0$ in $L^q(\Omega; w_1)$, $u_n \to u_0$ in $L^m(\Gamma; w_2)$, where u_0 is a non-negative critical point of $\Phi_{\lambda_0,\theta}$. To conclude that u_0 is a solution of problem $(P_{\lambda_0,\theta})$ it remains only to prove that $u_0 \not\equiv 0$. Since u_n and u_0 are critical points of $(\Phi_{\lambda_n,\theta})$ and $(\Phi_{\lambda_0,\theta})$, respectively, we have

$$\langle I'(u_n), u_n - u_0\rangle - \langle I'(u_0), u_n - u_0\rangle + \langle J'(u_n), u_n - u_0\rangle - \langle J'(u_0), u_n - u_0\rangle = J_{1,n} + J_{2,n},$$

where

$$J_{1,n} = \int_\Omega (f(\lambda_n, x, u_n) - f(\lambda_0, x, u_0))(u_n - u_0)\, dx,$$

$$J_{2,n} = \theta \int_\Gamma (g(x, u_n) - g(x, u_0))(u_n - u_0)\, d\Gamma\,.$$

It is easy to see that

$$0 \le \langle I'(u_n), u_n - u_0\rangle - \langle I'(u_0), u_n - u_0\rangle \le J_{1,n} + J_{2,n}. \tag{37}$$

Using (7) we get the estimate

$$|J_{1,n}| \le \tilde{C}_f \left(\lambda_n \int_\Omega |u_n|^{q-1}|u_n - u_0| w_1(x)\, dx + \lambda_0 \int_\Omega |u_0|^{q-1}|u_n - u_0| w_1(x)\, dx\right)$$

and it follows from the Hölder inequality that

$$|J_{1,n}| \le \tilde{C}_f \left(\sup_{n\ge 1} \lambda_n \|u_n\|^{q-1}_{q,\Omega,w_1} + \lambda_0 \|u_0\|^{q-1}_{q,\Omega,w_1}\right) \|u_n - u_0\|_{q,\Omega,w_1} \to 0. \tag{38}$$

By (26) and Hölder's inequality we find

$$|J_{2,n}| \le |\theta| \|N_g(u_n) - N_g(u_0)\|_{m/(m-1),\Gamma,w_2^{1/(1-m)}} \|u_n - u_0\|_{m,\Gamma,w_2} \to 0\,. \tag{39}$$

Relations (37), (38) and (39) yield

$$\langle I'(u_n), u_n - u_0\rangle - \langle I'(u_0), u_n - u_0\rangle \to 0 \quad \text{as } n \to \infty.$$

We show that $\|u_n - u_0\|_b \to 0$ as $n \to \infty$. We distinguish two cases which may occur

Case 1: $p \ge 2$. Using (28) we obtain

$$\|u_n - u_0\|_b^p \le C\left(\langle I'(u_n), u_n - u_0\rangle - \langle I'(u_0), u_n - u_0\rangle\right) \to 0 \quad \text{as } n \to \infty.$$

which shows that $\|u_n\|_b \to \|u_0\|_b$ as $n \to \infty$.

Case 2: $1 < p < 2$. We observe that it is enough to show that

$$\|u_n - u_0\|_b^2 \le C' \left(\langle I'(u_n), u_n - u_0\rangle - \langle I'(u_0), u_n - u_0\rangle\right) (\|u_n\|_b^{2-p} + \|u_0\|_b^{2-p}). \tag{40}$$

In order to prove (40) we recall the following result: for all $s > 0$ there is a constant $C_s > 0$ such that

$$(x + y)^s \le C_s(x^s + y^s) \qquad \text{for any } x, y \in (0, \infty). \tag{41}$$

Then we obtain

$$\begin{aligned} \|u_n - u_0\|_b^2 = &\left(\int_\Omega a(x)|Du_n - Du_0|^p\, dx + \int_\Gamma b(x)|u_n - u_0|^p\, d\Gamma\right)^{2/p} \le \\ &C_p \left[\left(\int_\Omega a(x)|Du_n - Du_0|^p\, dx\right)^{2/p} + \left(\int_\Gamma b(x)|u_n - u_0|^p\, d\Gamma\right)^{2/p}\right]. \end{aligned} \tag{42}$$

Using (29), (41) and the Hölder inequality we find

$$\int_\Omega a(x)|Du_n - Du_0|^p\,dx = \int_\Omega a(x)(|Du_n - Du_0|^2)^{p/2}\,dx \le$$

$$c_1 \int_\Omega \Big(a(|Du_n|^{p-2}Du_n - |Du_0|^{p-2}Du_0)\cdot(Du_n - Du_0)\Big)^{p/2} (a(|Du_n| + |Du_0|)^p)^{(2-p)/2}\,dx \le$$

$$c_1 \left(\int_\Omega a(|Du_n|^{p-2}Du_n - |Du_0|^{p-2}Du_0)\cdot(Du_n - Du_0)\,dx\right)^{p/2} \left(\int_\Omega a(|Du_n| + |Du_0|)^p\,dx\right)^{(2-p)/2} \le$$

$$c_2 \left(\int_\Omega a(|Du_n|^{p-2}Du_n - |Du_0|^{p-2}Du_0)(Du_n - Du_0)\,dx\right)^{p/2} \left(\int_\Omega (a|Du_n|^p + a|Du_0|^p)\,dx\right)^{(2-p)/2} \le$$

$$c_3 \left(\int_\Omega a(x)(|Du_n|^{p-2}Du_n - |Du_0|^{p-2}Du_0)\cdot(Du_n - Du_0)\,dx\right)^{p/2} (\|u_n\|_b^p + \|u_0\|_b^p)^{(2-p)/2} \le$$

$$c_4 \left(\int_\Omega a(x)(|Du_n|^{p-2}Du_n - |Du_0|^{p-2}Du_0)\cdot(Du_n - Du_0)\,dx\right)^{p/2} \left(\|u_n\|_b^{(2-p)p/2} + \|u_0\|_b^{(2-p)p/2}\right).$$

Using the last inequality and (41) we have the estimate

$$\left(\int_\Omega a(x)|Du_n - Du_0|^p\,dx\right)^{2/p} \le c_p\left(\langle I'(u_n), u_n - u_0\rangle - \langle I'(u_0), u_n - u_0\rangle\right)(\|u_n\|_b^{2-p} + \|u_0\|_b^{2-p}). \quad (43)$$

In a similar way we can obtain the estimate

$$\left(\int_\Gamma b(x)|u_n - u_0|^p\,d\Gamma\right)^{2/p} \le c_p'\left(\langle I'(u_n), u_n - u_0\rangle - \langle I'(u_0), u_n - u_0\rangle\right)(\|u_n\|_b^{2-p} + \|u_0\|_b^{2-p}). \quad (44)$$

It is now easy to observe that inequalities (42), (43) and (44) imply the estimate (40).

In both cases, by Corollary 1, $u_0 \not\equiv 0$. This concludes our proof. □

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Existence theorems for some classes of boundary value problems involving the p-Laplacian

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ABSTRACT. We prove an alternative for a nonlinear eigenvalue problem involving the p-Laplacian and study a subcritical boundary value problem for the same operator. The theoretical approach is the Mountain Pass Lemma and one of its variants (due to the first author), which is very useful in the study of eigenvalue problems.
KEY WORDS: p-Laplacian, nonlinear eigenvalue problem, critical point theory.
AMS SUBJECT CLASSIFICATION: 35 P 30, 47 H 15, 58 E 05.

For any fixed real number $p \in (1, +\infty)$ the p-Laplacian is defined by

$$\Delta_p u = \operatorname{div}\left(\mid \nabla u \mid^{p-2} \nabla u\right).$$

This operator appears in a variety of physical fields. For example, applications of Δ_p have been seen in Fluid Dynamics. The equation governing the motion of a fluid involves the p-Laplacian. More exactly the shear stress $\vec{\tau}$ and the velocity gradient ∇u of the fluid are related in the manner that

$$\vec{\tau}(x) = r(x) \mid \nabla u \mid^{p-2} \nabla u\,,$$

where $p = 2$ (resp., $p < 2$ or $p > 2$) if the fluid is Newtonian (resp., pseudoplastic or dilatant). Other applications of the p-Laplacian also appear in the study of flow through porous media ($p = \frac{3}{2}$), Nonlinear Elasticity ($p \geq 2$), or Glaciology ($1 < p \leq \frac{4}{3}$).

Throughout this paper, Ω stands for a bounded domain in $\mathbf{R}^N$. In the first section we are concerned with the following nonlinear eigenvalue problem with Dirichlet boundary condition and constraints on eigenvalues:

$$\left\{\begin{array}{ll} -\Delta_p u & = \lambda f(x,u), \quad \text{in } \Omega\,, \\ \quad u & = 0, \quad \text{on } \partial\Omega\,, \\ 0 < \lambda & \leq a\,, \end{array}\right. \tag{1}$$

where $a > 0$ is a given constant. The function f is supposed to satisfy

(H_1) f is a Carathéodory function, i.e., measurable in $x \in \Omega$ and continuous in $u \in \mathbf{R}$, with $f(x,0) \neq 0$ on a subset of Ω of positive measure;

(H_2) $\mid f(x,u) \mid \leq C_1 + C_2 \mid u \mid^{q-1}$, for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$, with constants $C_1 \geq 0$, $C_2 \geq 0$ and $1 < p \leq q < p^\star$, where

$$p^\star = \begin{cases} \dfrac{Np}{N-p}, & \text{if } p < N, \\ +\infty, & \text{if } p \geq N; \end{cases}$$

(H_3) there are constants $b_1 \geq 0, b_2 \geq 0, 1 \leq \gamma < p < \nu$ such that, for a.e. $x \in \Omega$ and every $u \in \mathbf{R}$,

$$f(x,u)u - \nu \int_0^u f(s,\tau)d\tau \geq -b_1 - b_2 \mid u \mid^\gamma .$$

By the Sobolev embedding Theorem, there exists a constant $C > 0$ such that, for every $u \in W_0^{1,p}(\Omega)$,

(2) $$\|u\|_{L^q}^q \leq C \|u\|_{W_0^{1,p}}^q .$$

For a later use we denote

(3) $$a_1 = c_1 \mid \Omega \mid^{(q-1)/q} \quad \text{and} \quad a_2 = C(c_1 \mid \Omega \mid^{(q-1)/q} + c_2 q^{-1}) .$$

Our approach relies on the following version of the celebrated Mountain Pass Theorem of Ambrosetti-Rabinowitz (see [1], [6]):

Lemma 1.([5]) *Let X be a Banach space and let $F : X \times \mathbf{R} \to \mathbf{R}$ be a C^1 functional verifying the hypotheses*

(a) there exist constants $\rho > 0$ and $\alpha > 0$ provided $F(v,\rho) \geq \alpha$, for every $v \in X$;

(b) there is some $r > \rho$ with $F(0,0) = F(0,r) = 0$.

Then the number

$$c := \inf_{g \in \Gamma} \max_{0 \leq \tau \leq 1} F(g(\tau)) ,$$

where

$$\Gamma = \{g \in C([0,1], X \times \mathbf{R}); g(0) = (0,0), g(1) = (0,r)\} ,$$

is a critical value of F.

Let us now state our main result concerning the eigenvalue problem (1). We shall keep the notations given in (2), (3) and, for simplicity, we use in the sequel $\|\cdot\|$ in place of $\|\cdot\|_{W_0^{1,p}}$.

Theorem 1. *Assume that the function $f : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies conditions (H_1)-(H_3). Let $\beta \in C^1(\mathbf{R}, \mathbf{R})$ be a function such that, for some constants $0 < \rho < r$, $\sigma > 0$, the following properties hold:*

(β_1) $\beta(0) = \beta(r) = 0$;

(β_2) $\rho^{\sigma+1} \geq q a_2$ and $\dfrac{\sigma+1}{q}\beta(\rho) = a_1$;

(β_3) $\lim_{|t| \to \infty} \beta(t) = +\infty$;

(β_4) $\beta'(t) < 0$ if and only if $t < 0$ or $\rho < t < r$.

Then, for each $a > 0$, the following alternative holds:
either

(i) $a > 0$ is an eigenvalue in problem (1) with a corresponding eigenfunction $u \in W_0^{1,p}(\Omega)$ located by

$$\alpha \le -\int_\Omega \int_0^{u(x)} f(x,t)dtdx + \frac{1}{ap}\|u\|^p \le a_1 + \alpha$$

or

(ii) one can find a positive number s with

$$\rho < s < r\,, \tag{4}$$

which determines an eigensolution $(u, \lambda) \in W_0^{1,p}(\Omega) \times (0, a]$ of the problem (1) by the relations

$$\|u\| = s^{-\sigma/q}(-\beta'(s))^{1/q}\,, \tag{5}$$

$$\lambda^{-1} = a^{-1} + s^{(q+\sigma p)/q}(-\beta'(s))^{(q-p)/q}\,, \tag{6}$$

$$\alpha \le \frac{s^{q+1}}{q}\|u\|^q + \frac{\sigma+1}{q}\beta(s) - \int_\Omega \int_0^{u(x)} f(x,t)dtdx + \frac{1}{ap}\|u\|^p \le a_1 + \alpha\,. \tag{7}$$

In the second section of this paper we consider another problem related to the p-Laplacian operator:

$$\begin{cases} -\Delta_p u & = \lambda \mid u \mid^{p-2} u + \mid u \mid^{q-2} u, \quad \text{in } \Omega\,, \\ u & = 0, \quad \text{on } \partial\Omega\,, \\ u & \not\equiv 0, \quad \text{in } \Omega\,. \end{cases} \tag{8}$$

Our result on this problem is

Theorem 2. *If $\lambda < \lambda_1(-\Delta_p) := \inf\{\int_\Omega \mid \nabla u \mid^p; u \in W_0^{1,p}(\Omega)\,, u \ne 0\,, \|u\|_{L^p} = 1\}$ and $1 < p < q < p^\star$, then the problem (8) has a weak solution.*

The key argument in the proof is the Mountain-Pass Theorem in the following variant:

Ambrosetti-Rabinowitz Theorem. *Let X be a real Banach space and $F : X \to \mathbf{R}$ be a C^1-functional. Suppose that F satisfies the Palais-Smale condition and the following geometric assumptions:*

$$\begin{array}{l}\textit{there exist positive constants } R \textit{ and } c_0 \textit{ such that} \\ F(u) \ge c_0, \textit{ for all } u \in X \textit{ with } \|u\| = R\,;\end{array} \tag{9}$$

$$\begin{array}{l}F(0) < c_0 \textit{ and there exists} \\ v \in X \textit{ such that } \|v\| > R \textit{ and } F(v) < c_0\,.\end{array} \tag{10}$$

Then the functional F possesses at least a critical point.

1 Proof of Theorem 1

In order to set problem (1) in terms of Lemma 1 we introduce the functional $F \in C^1(W_0^{1,p}(\Omega) \times \mathbf{R})$ by

$$F(v,t) = \frac{|\, t \,|^{\sigma+1}}{q} \|v\|^q + \frac{\sigma+1}{q} \beta(t) - \int_\Omega \int_0^{v(x)} f(x,t) dt dx + \frac{1}{ap} \|v\|^p . \tag{11}$$

From (β_1) and (11) we derive that condition (b) of Lemma 1 is valid.

From (H_2), (2) and (3) we see that, for every $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}
&\int_\Omega \int_0^{v(x)} f(x,t) dt dx \\
&\leq c_1 \|v\|_{L^1} + c_2 q^{-1} \|v\|_{L^q}^q \\
&\leq c_1 \,|\, \Omega \,|^{(q-1)/q} \|v\|_{L^q} + c_2 q^{-1} \|v\|_{L^q}^q \\
&\leq c_1 \,|\, \Omega \,|^{(q-1)/q} + (c_1 \,|\, \Omega \,|^{(q-1)/q} + c_2 q^{-1}) \|v\|_{L^q}^q \\
&\leq c_1 \,|\, \Omega \,|^{(q-1)/q} + C(c_1 \,|\, \Omega \,|^{(q-1)/q} + c_2 q^{-1}) \|v\|^q \\
&= a_1 + a_2 \|v\|^q
\end{aligned} \tag{12}$$

Relations (11), (12) and (β_2) yield

$$F(v,\rho) \geq (\frac{\rho^{\sigma+1}}{q} - a_2) \|v\|^q + \frac{\sigma+1}{q} \beta(\rho) - a_1 \geq \alpha ,$$

for every $v \in W_0^{1,p}(\Omega)$. This shows that the requirement (a) of Lemma 1 is fulfilled.

We check now that F verifies the Palais-Smale condition. To this end, let (v_n, t_n) be a sequence in $W_0^{1,p}(\Omega) \times \mathbf{R}$ such that $F(v_n, t_n)$ is bounded and

$$F'(v_n,t_n) = (F_v(v_n,t_n), F_t(v_n,t_n)) \to 0, \quad \text{in } W^{-1,p'}(\Omega) \times \mathbf{R} ,$$

where $p' = \dfrac{p}{p-1}$. Therefore

$$|\, F(v_n,t_n) \,| \leq M \tag{13}$$

$$-F_v(v_n,t_n) = |\, t_n \,|^{\sigma+1} \|v_n\|^{q-p} \Delta_p v_n + f(\cdot, v_n) + a^{-1} \Delta_p v_n \to 0 \quad \text{in } W^{-1,p'}(\Omega) \tag{14}$$

$$F_t(v_n,t_n) = |\, t_n \,|^{\sigma} (\operatorname{sgn} t_n) \|v_n\|^q + \beta'(t_n) \to 0, \quad \text{in } \mathbf{R} . \tag{15}$$

From (11), (12) and (13) we infer that

$$M \geq (q^{-1} \,|\, t_n \,|^{\sigma+1} - a_2) \|v_n\|^q + \frac{\sigma+1}{q} \beta(t_n) - a_1 .$$

But, by condition (β_3), this shows that (t_n) is bounded in $\mathbf{R}$.

Without loss of generality we may assume that (v_n) is bounded away from 0. We treat separately two cases.

Firstly, assume that along a subsequence one has $t_n \to 0$. Then, by (β_4), it follows that $\beta'(t_n) \to \beta'(0) = 0$. So, by (15),

$$|\, t_n \,|^{\sigma} \; \|v_n\|^q \to 0, \quad \text{as } n \to \infty\,. \tag{16}$$

From (11), (13) and (16) we see that

$$\int_\Omega \int_0^{v_n(x)} f(x,\tau)d\tau dx - \frac{1}{ap}\,\|v_n\|^p \quad \text{is bounded in } \mathbf{R}\,. \tag{17}$$

Since $t_n \to 0$ and (v_n) is bounded away from zero it is clear from (16) that

$$\begin{aligned}
&|\, t_n \,|^{\sigma+1} \; \|v_n\|^{q-p}\| \; \|\Delta_p v_n\|_{W_0^{-1,p'}} \\
&=|\, t_n \,|\,|\, t_n \,|^{\sigma} \; \|v_n\|^{q-p} \; \|v_n\|^{p-1} \\
&=|\, t_n \,|\,|\, t_n \,|^{\sigma} \; \|v_n\|^{q} \; \|v_n\|^{-1} \to 0, \quad \text{as } n \to \infty\,.
\end{aligned}$$

Thus, (14) implies

$$f(\cdot, v_n) + a^{-1}\,\Delta_p v_n \to 0, \quad \text{as } n \to \infty\,. \tag{18}$$

From (17) and (18) we find that, for some constant $M > 0$ and with $\nu > 2$ in (H_3),

$$\begin{aligned}
&M + \nu^{-1}\|v_n\| \\
&\geq \frac{1}{ap}\,\|v_n\|^p - \int_\Omega \int_0^{v_n(x)} f(x,\tau)d\tau dx \\
&+\tfrac{1}{\nu}\,(\int_\Omega f(x,v_n)v_n dx + a^{-1}\int_\Omega (\Delta_p v_n)v_n dx) \\
&= \frac{1}{a}\,(\frac{1}{p} - \frac{1}{\nu})\,\|v_n\|^p + \frac{1}{\nu}\int_\Omega (f(x,v_n)v_n - \nu \int_0^{v_n(x)} f(x,\tau)d\tau)dx\,,
\end{aligned}$$

if n is sufficiently large. Then hypothesis (H_3) and inequality (2) ensure us that some new constants $d_1 \geq 0$ and $d_2 \geq 0$ exist such that

$$\begin{aligned}
&M + \nu^{-1}\|v_n\| \\
&\geq \frac{1}{a}\,(\frac{1}{p} - \frac{1}{\nu})\,\|v_n\|^p - \frac{1}{\nu}(b_1 \,|\, \Omega \,| + b_2\|v_n\|^{\gamma}_{L^\gamma}) \\
&\geq \frac{1}{a}\,(\frac{1}{p} - \frac{1}{\nu})\,\|v_n\|^p - d_1 - d_2\,\|v_n\|^{\gamma}\,.
\end{aligned}$$

Recalling that $1 \leq \gamma < p < \nu$, the last estimate shows that (v_n) is bounded in $W_0^{1,p}(\Omega)$. On the other hand, the growth condition in (H_2) ensures that the restriction of Nemytskii's operator to $W_0^{1,p}(\Omega)$, namely,

$$v \in W_0^{1,p}(\Omega) \longmapsto f(\cdot, v(\cdot)) \in W^{-1,p'}(\Omega)\,,$$

is a compact mapping, in the sense that it maps any bounded set onto a relatively compact one (see, for details, de Figueiredo [3] or Rabinowitz [6]). Thus, passing eventually to a subsequence,

$$f(\cdot, v_n(\cdot)) \quad \text{converges in } W_0^{-1,p'}(\Omega)\,. \tag{19}$$

By (18) and (19) we conclude that (v_n) possesses a convergent subsequence in $W_0^{1,p}(\Omega)$.

Assume now that (t_n) is bounded away from 0. Then, by (15), we see that (v_n) is bounded in $W_0^{1,p}(\Omega)$. Hence (19) holds. From (14) it follows that

$$(1 + a \mid t_n \mid^{\sigma+1} \|v_n\|^{q-p})\, \Delta_p v_n \quad \text{is convergent in } W^{-1,p'}(\Omega)\,,$$

which shows that $(\Delta_p v_n)$ converges in $W^{-1,p'}(\Omega)$. Finally, we obtain that, up to a subsequence, (v_n) converges in $W_0^{1,p}(\Omega)$. This concludes the verification of the Palais-Smale condition for the functional F.

The hypotheses of Lemma 1 are now verified. Thus, there exists a point $(u, s) \in W_0^{1,p}(\Omega) \times \mathbf{R}$ satisfying

$$-\Delta_p u = \frac{1}{a^{-1} + \mid s \mid^{\sigma+1} \|u\|^{q-p}}\, f(\cdot, u)\,; \tag{20}$$

$$\mid s \mid^{\sigma} \;(\operatorname{sgn} s)\, \|u\|^q + \beta'(s) = 0\,; \tag{21}$$

$$\frac{\mid s \mid^{\sigma+1}}{q} \|u\|^q + \frac{\sigma+1}{q} \beta(s) - \int_\Omega \int_0^{u(x)} f(x,t)\,dt\,dx + \frac{1}{ap} \|u\|^p \geq \alpha\,. \tag{22}$$

From (21) we observe that

$$s\beta'(s) \leq 0\,. \tag{23}$$

There are two cases:

Case 1: $s = 0$. Then the assertion (i) in the alternative of Theorem 1 is deduced from (20) and (22). The last inequality of (i) is obtained from the definition of c and Γ in Lemma 1, making use of the path $g \in \Gamma$ given by $g(t) = (0, tr)$, for $0 \leq t \leq 1$.

Case 2: $s \neq 0$. We argue by contradiction. If $s < 0$ then, by (β_4), it follows that $\beta'(s) < 0$, which contradicts (23). So, the only possibility is $s > 0$. Using (β_4) again it turns out

$$\rho \leq t \leq r\,. \tag{24}$$

If $t = \rho$ or $t = r$, relation (21) and assumption (β_4) imply $u = 0$. This leads to a contradiction between (20) and our hypothesis (H_1). We proved that (24) reduces to (4). Since $s > 0$, (21) gives rise to (5). From (20) it is clear that $(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbf{R}$ is an eigensolution of (1), where

$$\lambda = \frac{1}{a^{-1} + s^{\sigma+1} \|u\|^{q-p}}\,. \tag{25}$$

Substituting $\|u\|$ as determined by (5) in (25) we arrive at (6). The first inequality of (7) is just (22). The second inequality of (7) follows from Lemma 1, by choosing the path $g(t) = (0, tr)$, $0 \leq t \leq 1$. ∎

Corollary 1. *Assume that the function $f : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies hypotheses (H_1)-(H_3) and let $a > 0$ be a number which is not an eigenvalue of the problem (1). Then there exists a sequence $(u_n, \lambda_n) \in W_0^{1,p}(\Omega) \times (0, a)$ of eigensolutions of (1) with the properties*

$$u_n \to 0 \ \text{ in } W_0^{1,p}(\Omega), \ \lambda_n \to 0 \ \text{ and } \ \lambda_n^{-1}\|u_n\|^p \to 0, \quad \text{as } n \to \infty .$$

Proof. For every $\varepsilon > 0$ one can find $\beta_\varepsilon \in C^1(\mathbf{R}, \mathbf{R})$ satisfying (β_1)-(β_4) with $\rho = \rho_\varepsilon < r = r_\varepsilon$, which depends on ε, and $\sigma > 0$, $\alpha > 0$ independent of ε such that

$$|\ \beta_\varepsilon'(t)\ | \leq \varepsilon^q t^{-1}, \quad \text{for every } \ t \geq (qa_2)^{1/(\sigma+1)} . \tag{26}$$

Applying Theorem 1, one obtains the number $s = s_\varepsilon \in (\rho_\varepsilon, r_\varepsilon)$ that describes an eigensolution $(u_\varepsilon, \lambda_\varepsilon)$ of (1) by equalities (5) and (6) with $u = u_\varepsilon$ and $\lambda = \lambda_\varepsilon$. Clearly, we can assume

$$s_\varepsilon \to +\infty , \quad \text{as } \varepsilon \to 0 . \tag{27}$$

Hence, by (5), (26) and (27), we infer that

$$\|u_\varepsilon\| = s_\varepsilon^{-\sigma/q}\,(-\beta'(s_\varepsilon))^{1/q} \leq \varepsilon s_\varepsilon^{-(\sigma+1)/q} \to 0 , \quad \text{as } \varepsilon \to 0 . \tag{28}$$

We know that the following equality holds

$$-\frac{1}{\lambda_\varepsilon}\,\Delta_p u_\varepsilon = f(x, u_\varepsilon) .$$

Letting $\varepsilon \to 0$ we notice that, in view of (H_1) and $u_\varepsilon \to 0$ in $W_0^{1,p}(\Omega)$, it follows that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. In addition, we get from (6) that

$$(\lambda_\varepsilon^{-1} - a^{-1})^q = s_\varepsilon^{q+\sigma p}(-\beta'(s_\varepsilon))^{q-p} \leq \varepsilon^{q(q-p)} s_\varepsilon^{(\sigma+1)p} . \tag{29}$$

By (28) and (29) we observe that

$$\|u_\varepsilon\|^p\,(\lambda_\varepsilon^{-1} - a^{-1}) \leq \varepsilon^q ,$$

which implies, taking into account (28), that

$$\lambda_\varepsilon^{-1}\,\|u_\varepsilon\|^p \to 0 , \quad \text{as } \varepsilon \to 0 .$$

This completes our proof. ∎

Corollary 2. *Under the hypotheses of Corollary 1, for every function $\beta \in C^1(\mathbf{R}, \mathbf{R})$ satisfying conditions (β_1)-(β_4) with fixed constants ρ, r, σ, α, there is a one-to-one mapping from $[1, +\infty)$ into the set of eigensolutions (u, λ) of the problem (1). In particular, there exist uncountable many solutions (u, λ) of (1).*

Proof. Notice that if $\beta \in C^1(\mathbf{R}, \mathbf{R})$ satisfies the requirements (β_1)-(β_4) for given numbers ρ, r, σ, α, then this is true for each function $\delta\beta$, with an arbitrary number $\delta \geq 1$. We may suppose that there is some $a > 0$ which is not an eigenvalue of (1). Applying Theorem 1 with

$\delta\beta$, for $\delta \geq 1$, in place of δ, one finds an eigensolution $(u_\delta, \lambda_\delta) \in W_0^{1,p}(\Omega) \times (0, a)$ and a number $s_\delta \in (\rho, r)$ such that

$$\|u_\delta\| = s_\delta^{-\sigma/q} \left(-\beta'(s_\delta)\right)^{1/q} \delta^{1/q} \tag{30}$$

and, by (25),

$$\lambda_\delta^{-1} = a^{-1} + s_\delta^{\sigma+1} \|u_\delta\|^{q-p} . \tag{31}$$

Let $\delta_1, \delta_2 \geq 1$ with $\delta_1 \neq \delta_2$. Then (31) shows that $s_{\delta_1} = s_{\delta_2}$. Thus (30) yields $\delta_1 = \delta_2$. This contradiction completes the proof. ■

In some situations the qualitative informations provided by Theorem 1 and Corollaries 1 and 2 can be improved by direct methods in studying the eigenvalue problem (1).

Example 1. Assume that the Carathéodory function $f : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies (H_1) and the growth condition

$$\left| \int_0^t f(x,\tau)d\tau \right| \leq C_1 + C_2 \mid t \mid^p, , \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbf{R}, \tag{32}$$

with constants $C_1 \geq 0$ and $C_2 \geq 0$. Using the constant $C > 0$ entering in (2), with $q = p$, we check that every number $\lambda > 0$ which satisfies

$$\lambda < \lambda^\star := \frac{1}{pCC_2} \tag{33}$$

is an eigenvalue of the boundary value problem

$$\begin{cases} -\Delta_p u &= \lambda f(x,u), \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega . \end{cases}$$

In order to justify this, corresponding to each λ in (33) we introduce the functional I_λ : $W_0^{1,p}(\Omega) \to \mathbf{R}$ by

$$I_\lambda(v) = -\int_\Omega \int_0^{v(x)} f(x,t)dtdx + \frac{1}{\lambda p} \|v\|^p .$$

The assumption (32) allows us to write

$$\begin{aligned} I_\lambda(v) &\geq \tfrac{1}{\lambda p} \|v\|^p - C_1 \mid \Omega \mid - C_2 \|v\|_{L^p}^p \\ &\geq \left(\tfrac{1}{\lambda p} - CC_2\right) \|v\|^p - C_1 \mid \Omega \mid, \end{aligned} \tag{34}$$

for every $v \in W_0^{1,p}(\Omega)$. From (33) and (34) it follows that the functional I_λ is bounded from below, coercive and (sequentially) weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. Therefore the infimum of I_λ is achieved at some $u \in W_0^{1,p}(\Omega)$ which solves the above boundary value problem corresponding to any λ in (33).

Remark 1. The problem treated in Example 1 covers the assymptotically linear case discussed by Brezis&Nirenberg [2] and Mironescu&Rădulescu [4]. In these references sharp information is given concerning the solvability (unsolvability) of the cases outside (33). In the situation where (33) holds with p replaced by $\sigma < p$, the considered eigenvalue problem admits every positive number λ as an eigenvalue.

2 Proof of Theorem 2

Our hypothesis

$$\lambda < \lambda_1(-\Delta_p) := \inf_{W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}$$

implies the existence of some $C_0 > 0$ such that, for every $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega (|\nabla v|^p - \lambda |v|^p) dx \geq C_0 \int_\Omega |\nabla v|^p \, dx \, . \tag{35}$$

Set

$$g(u) = \begin{cases} u^{q-1}, & \text{if } u \geq 0\,, \\ 0, & \text{if } u < 0 \end{cases}$$

and $G(u) = \int_o^u g(t)dt$. Denote

$$F(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda |u|^p) dx - \int_\Omega G(u) dx \, .$$

Observe that

$$|G(u)| \leq C \; |u|^q$$

and, by our hypothesis $1 < p < q < p^\star$, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$, which implies that F is well defined on $W_0^{1,p}(\Omega)$.

A straightforward computation shows that F is a C^1 function and, for every $v \in W_0^{1,p}(\Omega)$,

$$F'(u)(v) = \int_\Omega (|\nabla u|^{p-2} \, \nabla u \cdot \nabla v - \lambda \; |u|^{p-2} \, uv) dx - \int_\Omega g(u) v dx \, .$$

We prove in what follows that F satisfied the hypotheses of the Mountain-Pass Theorem.

Verification of (9): We may write, for every $u \in \mathbf{R}$,

$$|g(u)| \leq |u|^{q-1} \; .$$

Thus, for every $u \in \mathbf{R}$,

$$|G(u)| \leq \frac{1}{q} \; |u|^q \; . \tag{36}$$

Now, by (36) and the Sobolev embedding Theorem,

$$F(u) \geq C_0 \, \|u\|^p - \frac{1}{q} \, \|u\|^q \, , \tag{37}$$

for every $u \in W_0^{1,p}(\Omega)$.

For $\varepsilon > 0$ and $R > 0$ small enough, we deduce by (36) that, for every $u \in W_0^{1,p}(\Omega)$ with $\|u\| = R$,

$$F(u) \geq c_0 > 0 .$$

Verification of (10): Choose $u_0 \in W_0^{1,p}(\Omega)$, $u_0 > 0$ in Ω. Then, by $1 < p < q < p^\star$, it follows that if $t > 0$ is large enough,

$$F(tu_0) = \frac{t^p}{p} \int_\Omega (\mid \nabla u_0 \mid^p - \lambda \mid u_0 \mid^p) dx - t^q \int_\Omega u_0^q dx < 0 .$$

Verification of the Palais-Smale condition: Let (u_n) be a sequence in $W_0^{1,p}(\Omega)$ such that

(38) $$\sup_n \mid F(u_n) \mid < +\infty ,$$

(39) $$\|F'(u_n)\|_{W^{-1,p'}} \to 0 , \quad \text{as } n \to \infty .$$

We prove firstly that (u_n) is bounded in $W_0^{1,p}(\Omega)$. Remark that (39) implies that, for every $v \in W_0^{1,p}(\Omega)$,

(40) $$\begin{aligned} &\int_\Omega (\mid \nabla u_n \mid^{p-2} \nabla u_n \cdot \nabla v - \lambda \mid u_n \mid^{p-2} u_n v) dx \\ &= \int_\Omega g(u_n) v dx + o(1) \|v\| , \quad \text{as } n \to \infty . \end{aligned}$$

Choosing $v = u_n$ in (40) we find

(41) $$\int_\Omega (\mid \nabla u_n \mid^p - \lambda \mid u_n \mid^p) dx = \int_\Omega g(u_n) u_n dx + o(1) \|u_n\| .$$

Remark that (38) means that there exists $M > 0$ such that, for any $n \geq 1$,

(42) $$\mid \frac{1}{p} \int_\Omega (\mid \nabla u_n \mid^p - \lambda \mid u_n \mid^p) dx - \int_\Omega G(u_n) dx \mid \leq M .$$

But a simple computation yields

(43) $$\int_\Omega g(u_n) u_n dx = q \int_\Omega G(u_n) dx .$$

Combining (41), (42) and (43) we find

(44) $$\alpha \int_\Omega G(u_n) dx = O(1) + o(1) \|u_n\| ,$$

where $\alpha = q - p > 0$. Thus, by (41) and (44),

$$\|u_n\|^p \leq O(1) + o(1) \|u_n\| ,$$

which means that $\|u_n\|$ is bounded.

It remains to prove that (u_n) is relatively compact. We consider the case $p < N$. First of all we remark that (40) may be written

$$\int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \cdot \nabla v dx = \int_\Omega h(u_n) v dx + o(1) \|v\| , \tag{45}$$

for every $v \in W_0^{1,p}(\Omega)$, where

$$h(u) = g(u) + \lambda \mid u \mid^{p-2} u .$$

Obviously, h is continuous and there exists $C > 0$ such that

$$\mid h(u) \mid \leq C \, (1 + \mid u \mid^{(Np-N+p)/(N-p)}) . \tag{46}$$

Moreover

$$h(u) = o(\mid u \mid^{Np/(N-p)}) , \quad \text{as } \mid u \mid \to \infty . \tag{47}$$

Observing that $(-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$ is a continuous operator, it follows by (45) that it suffices to show that $h(u_n)$ is relatively compact in $W^{-1,p'}(\Omega)$. By Sobolev's Theorem, this will be achieved by proving that a subsequence of $h(u_n)$ is convergent in $(L^{(Np)/(N-p)}(\Omega))^\star = L^{(Np)/(Np-N+p)}(\Omega)$.

Since (u_n) is bounded in $W_0^{1,p}(\Omega) \subset L^{(Np)/(N-p)}(\Omega)$ we can suppose that, up to a subsequence,

$$u_n \to u \in L^{(Np)/(N-p)}(\Omega) , \quad \text{a.e. in } \Omega .$$

Moreover, by Egorov's Theorem, for each $\delta > 0$, there exists a subset A of Ω with $\mid A \mid < \delta$ and such that

$$u_n \to u , \quad \text{uniformly in } \Omega \setminus A .$$

So, it is sufficient to show that

$$\int_A \mid h(u_n) - h(u) \mid^{(Np)/(Np-N+p)} dx \leq \eta ,$$

for any fixed $\eta > 0$. But, by (46),

$$\int_A \mid h(u) \mid^{(Np)/(Np-N+p)} dx \leq C \int_A (1 + \mid u \mid^{(Np)/(N-p)}) dx ,$$

which can be made arbitrarily small if we choose a sufficiently small $\delta > 0$.

We have, by (47),

$$\int_A \mid h(u_n) - h(u) \mid^{(Np)/(Np-N+p)} dx \leq \varepsilon \int_A \mid u_n - u \mid^{(Np)/(N-p)} dx + C_\varepsilon \mid A \mid ,$$

which can be also made arbitrarily small, by Sobolev's Theorem and by the boundedness of (u_n) in $W_0^{1,p}(\Omega)$.

Hence, F satisfies Palais-Smale Condition and, by Ambrosetti-Rabinowitz Theorem, the problem (8) has a weak solution. ■

Remark 2. We are not able to decide at this stage what happens if $\lambda > \lambda_1(-\Delta_p)$. The main difficulty consists in the impossibility of defining in a suitable manner the orthogonal of a set, so to split the Banach space $W_0^{1,p}(\Omega)$, $p \neq 2$, as a direct sum of its first eigenspace and the corresponding orthogonal. A more general version of Theorem 2 can be obtained by replacing the term $\mid u \mid^{q-2} u$ in (8) by a function f(x,u) whose behaviour at $u = 0$ and for $\mid u \mid \to +\infty$ is similar to the one of $\mid u \mid^{q-2} u$. The final part of the proof of Theorem 2, that is, the deduction of the relative compactness of u_n from its boundedness, can also be derived using the continuity of Nemytskii's operator $u \longmapsto h(u)$ on $L^{p^\star}(\Omega)$.

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Chapitre II
Problèmes elliptiques singuliers: existence, unicité et explosion des solutions

1. F. Cîrstea et V. Rădulescu, Blow-up boundary solutions for semilinear elliptic problems, *Nonlinear Analysis, T.M.A.* **48** (2002), 541-554.

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Blow-up boundary solutions of semilinear elliptic problems

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Abstract. Let f be a non-decreasing C^1-function such that $f > 0$ on $(0,\infty)$, $f(0) = 0$ and $\int_1^\infty 1/\sqrt{F(t)}dt$ is finite, where $F(t) = \int_0^t f(s)\,ds$. We prove the existence of positive large solutions to the equation $\Delta u = p(x)f(u)$ in a smooth bounded domain $\Omega \subset \mathbf{R}^N$, $N \geq 3$, provided that p is a non-negative continuous function so that any of its zeros is surrounded by a surface strictly included in Ω on which p is positive. Under additional hypotheses on p we deduce the existence of maximal solutions if Ω is unbounded.

1 Introduction and the main results

We consider the following semilinear elliptic equation

$$\begin{cases} \Delta u = p(x)f(u) & \text{in } \Omega\,, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \Omega\,, \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ is a smooth domain (bounded or possibly unbounded) with compact (possibly empty) boundary. We assume throughout this paper that p is a non-negative function such that $p \in C^{0,\alpha}(\overline{\Omega})$ if Ω is bounded, and $p \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, otherwise. The non-linearity f is assumed to fulfill

$(f1)$ $\quad f \in C^1[0,\infty),\ f' \geq 0,\ f(0) = 0$ and $f > 0$ on $(0,\infty)$

and the Keller-Osserman condition

$(f2)$ $\quad \displaystyle\int_1^\infty [2F(t)]^{-1/2}\,dt < \infty\,, \quad \text{where} \quad F(t) = \int_0^t f(s)\,ds.$

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The main purpose of the paper is to find properties of *large solutions* of (1), that is solutions u satisfying $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ (if $\Omega \not\equiv \mathbf{R}^N$), or $u(x) \to \infty$ as $|x| \to \infty$ (if $\Omega = \mathbf{R}^N$). In the latter case the solution is called to be an *entire large* solution.

Problems of this type have been originally studied by Loewner and Nirenberg in their celebrated paper [11]. Their work deals with partial differential equations having a "partial conformal invariance" and is motivated by a concrete problem arising in Riemannian Geometry. More precisely, in [11] Loewner and Nirenberg proved the remarkable result that (1) has a maximal solution, provided that $\Omega \neq \mathbf{R}^N$, $p \equiv \text{Const.} > 0$ in Ω and $f(u) = u^{(N+2)/(N-2)}$.

In [1] and [12] it is considered problem (1) in the special case when Ω is bounded and $p > 0$ in $\overline{\Omega}$. More precisely, in [1] Bandle and Marcus described the precise asymptotic behavior of large solutions near the boundary and established the uniqueness of such solutions, while in [12] Marcus obtained existence results for large solutions.

The first result we obtain in this paper is an existence theorem for large solutions when Ω is bounded.

Theorem 1 *Suppose Ω is bounded and p satisfies*
($p1$) *for every $x_0 \in \Omega$ with $p(x_0) = 0$, there is a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega_0} \subset \Omega$ and $p > 0$ on $\partial\Omega_0$.*
Then problem (1) has a positive large solution.

This result generalizes Theorem 3.1 in Marcus [12] and Lemma 2.6 in Cheng-Ni [4] since condition ($p1$) is weaker than the assumption that $p > 0$ on $\partial\Omega$, as required in [4, Lemma 2.6] and in [12, Theorem 3.1]. Indeed, the continuity of p, the compactness of $\partial\Omega$ and the fact that $p > 0$ on $\partial\Omega$ imply the existence of some $\delta > 0$ such that $p > 0$ in

$$\Omega_\delta := \{x \in \overline{\Omega};\ \operatorname{dist}(x, \partial\Omega) \leq \delta\}.$$

Therefore, all the zeros of p are included in $\Omega_0 = \overline{\Omega} \setminus \Omega_\delta \subset\subset \Omega$. Hence $p > 0$ on $\partial\Omega_0$, so ($p1$) is fulfilled.

We now consider problem (1) when $\Omega = \mathbf{R}^N$, and first observe that any entire large solution of (1) is positive. Indeed, assume there exists $x_0 \in \mathbf{R}^N$ such that $u(x_0) = 0$. Since u is an entire large solution, we can choose $R > |x_0|$ such that $u > 0$ on $\partial B(0, R)$. Thus, by Theorem 5 in the Appendix, the problem

$$\begin{cases} \Delta\zeta = p(x)f(\zeta) & \text{in } B(0,R)\,, \\ \zeta = u & \text{on } \partial B(0,R)\,, \\ \zeta \geq 0 & \text{in } B(0,R) \end{cases}$$

has a unique solution, which is positive. By uniqueness, of course, $\zeta = u$, which is the required contradiction. This shows that u cannot vanish in $\mathbf{R}^N$.

The next purpose of the paper is to prove the existence of an entire maximal solution for (1), under more general hypotheses than in Cheng-Ni [4]. They investigate the structure of all positive solutions of (1) in the special case when $f(u) = u^\gamma$, $\gamma > 1$, and they also establish existence of the maximal classical solution U of (1), under the hypotheses that this equation possesses at least a positive entire solution and there is a sequence of smooth bounded domains $(\Omega_n)_{n\geq 1}$ such that, for any $n \geq 1$,

$$\overline{\Omega_n} \subseteq \Omega_{n+1}, \quad \mathbf{R}^N = \cup_{n=1}^{\infty} \Omega_n, \quad p > 0 \text{ on } \partial\Omega_n. \tag{2}$$

Cheng and Ni also proved in [4] that the maximal solution U is the unique entire large solution of problem (1), under the additional restriction that for some $l > 2$ there exist two positive constants C_1, C_2 such that

$$C_1 p(x) \leq |x|^{-l} \leq C_2 p(x) \quad \text{for large } |x|\,. \tag{3}$$

Our result in the case $\Omega = \mathbf{R}^N$ is the following

Theorem 2 *Assume that $\Omega = \mathbf{R}^N$ and that problem (1) has at least a solution. Suppose that p satisfies the condition*
$(p1)'$ There exists a sequence of smooth bounded domains $(\Omega_n)_{n\geq 1}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $\mathbf{R}^N = \cup_{n=1}^{\infty}\Omega_n$, and $(p1)$ holds in Ω_n, for any $n \geq 1$.

Then there exists a maximal classical solution U of (1).

If p verifies the additional condition

$$(p2) \qquad \int_0^\infty r\Phi(r)\,dr < \infty\,, \quad \text{where } \Phi(r) = \max\{p(x): \ |x| = r\},$$

then U is an entire large solution.

In view of the remark above that condition $(p1)$ on Ω is weaker than the requirement that $p > 0$ on $\partial\Omega$, it follows that condition $(p1)'$ is weaker than the assumption (2) required in [4], and also assumption $(p2)$ is weaker than condition (3) imposed in [4].

We now observe that if $p(x) > 0$ for $|x|$ sufficiently large, then $(p1)'$ is automatically satisfied. Therefore it is natural to ask us if there exists $p \geq 0$ which satisfies $(p2)$ and $(p1)'$, with p vanishing in every neighborhood of infinity. The answer is positive by the following example. Take

$$\begin{cases} p(r) = 0 \quad \text{for} \quad r = |x| \in [n-1/3, n+1/3],\ n \geq 1; \\ p(r) > 0 \quad \text{in} \quad \mathbf{R}_+ \setminus \bigcup_{n=1}^{\infty} [n-1/3, n+1/3]; \\ p \in C^1[0,\infty) \quad \text{and} \quad \max\limits_{r\in[n,n+1]} p(r) = \dfrac{2}{n^2(2n+1)}. \end{cases}$$

Of course, $(p1)'$ is fulfilled, for $\Omega_n = B(0, n+1/2)$. On the other hand, condition $(p2)$ is also satisfied since

$$\int_1^\infty r\Phi(r)\,dr = \sum_{n=1}^\infty \int_n^{n+1} rp(r)\,dr \le \sum_{n=1}^\infty \int_n^{n+1} \frac{2}{n^2(2n+1)} r\,dr = \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

We now consider the case in which $\Omega \neq \mathbf{R}^N$ and Ω is unbounded; we say that a large solution u of (1) is *regular* if u tends to zero at infinity. In [12, Theorem 3.1] Marcus proved for this case the existence of regular large solutions to problem (1) by assuming that there exist $\gamma > 1$ and $\beta > 0$ such that

$$\liminf_{t\to 0} f(t)t^{-\gamma} > 0 \qquad \text{and} \qquad \liminf_{|x|\to\infty} p(x)|x|^\beta > 0.$$

The large solution constructed in Marcus [12] is the smallest large solution of problem (1). In the next result we show that problem (1) admits a maximal classical solution U and that U blows-up at infinity if $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$.

Theorem 3 *Suppose that $\Omega \neq \mathbf{R}^N$ is unbounded and that problem (1) has at least a solution. Assume that p satisfies condition $(p1)'$ in Ω. Then there exists a maximal classical solution U of problem (1).*

If $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$ and p satisfies the additional condition $(p2)$, with $\Phi(r) = 0$ for $r \in [0, R]$, then the maximal solution U is a large solution that blows-up at infinity.

In conclusion, by Theorem 3 and the recalled result of Marcus, in the case $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$, problem (1) admits large solutions tending to zero or to infinity as $|x| \to \infty$ (regular or normal large solutions).

In Section 2 we prove Theorem 1, while in Section 3 we prove Theorems 2 and 3. In Section 4 we prove the following necessary condition for the existence of entire large solutions to equation (1) if p satisfies $(p2)$, and for which f is not assumed to satisfy $(f2)$, and p is not required to be so regular as before. More precisely, we prove

Theorem 4 *Assume that $p \in C(\mathbf{R}^N)$ is a non-negative and non-trivial function which satisfies $(p2)$. Let f be a function satisfying assumption $(f1)$. Then condition*

$$\int_1^\infty \frac{dt}{f(t)} < \infty \tag{4}$$

is necessary for the existence of entire large solutions to (1).

2 Existence results for bounded domains

Lemma 1 *Assume that conditions* $(f1)$ *and* $(f2)$ *are fulfilled. Then*

$$\int_1^\infty \frac{dt}{f(t)} < \infty\,.$$

Proof. Fix $R > 0$ and denote $B = B(0,R)$. By Theorem 5 in the Appendix the boundary value problem

$$\begin{cases} \Delta u_n = f(u_n), & \text{in } B \\ u_n = n, & \text{on } \partial B \\ u_n \geq 0,\ u_n \not\equiv 0, & \text{in } B \end{cases} \tag{5}$$

has a unique positive solution. Since f is non-decreasing, it follows by the maximum principle that $u_n(x)$ increases with n, for any fixed $x \in B$.

We first claim that (u_n) is uniformly bounded in every compact subdomain of B. Indeed, let $K \subset B$ be any compact set and $d := \operatorname{dist}(K, \partial B)$. Then

$$0 < d \leq \operatorname{dist}(x, \partial B), \quad \forall x \in K. \tag{6}$$

By Proposition 1 of Bandle-Marcus [1], there exists a continuous, non-increasing function $\mu : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$u_n(x) \leq \mu(\operatorname{dist}(x, \partial B)), \quad \forall x \in K.$$

The claim now follows from (6). Thus, for every $x \in B$ we can define $u(x) := \lim_{n\to\infty} u_n(x)$.

We next show that u is a classical large solution of

$$\Delta u = f(u) \quad \text{in } B. \tag{7}$$

Fix $x_0 \in B$ and let $r > 0$ be such that $\overline{B(x_0, r)} \subset B$. Let $\Psi \in C^\infty(B)$ be such that $\Psi \equiv 1$ in $\overline{B(x_0, r/2)}$ and $\Psi \equiv 0$ in $B \setminus B(x_0, r)$. We have

$$\Delta(\Psi u_n) = 2\nabla\Psi \cdot \nabla u_n + p_n,$$

where $p_n = u_n \Delta\Psi + \Psi \Delta u_n$. Since (u_n) is uniformly bounded on $\overline{B(x_0, r)}$ and f is non-decreasing on $[0, \infty)$, it follows that $\|p_n\|_\infty \leq C$, where C is a constant independent of n. From now on, using the same argument given in the proof of Lemma 3 of [9], we find that (u_n) converges in $C^{2,\alpha}(\overline{B(x_0, r_1)})$, for some $r_1 > 0$. Since $x_0 \in B$ is arbitrary, this shows that $u \in C^2(B)$ and u is a positive solution of (7). Moreover, by the Gidas-Ni-Nirenberg theorem in [6], u is radially symmetric in B, namely $u(x) = u(r)$, $r = |x|$, and u satisfies in the r variable the equation

$$u''(r) + \frac{N-1}{r} u'(r) = f(u(r)), \qquad 0 < r < R.$$

This equation can be rewritten as follows

$$(r^{N-1}u'(r))' = r^{N-1}f(u(r)), \qquad 0 < r < R. \tag{8}$$

Integrating (8) from 0 to r we obtain

$$u'(r) = r^{1-N}\int_0^r s^{N-1}f(u(s))\,ds, \qquad 0 < r < R.$$

Hence u is a non-decreasing function and

$$u'(r) \le r^{1-N}f(u(r))\int_0^r s^{N-1}\,ds = \frac{r}{N}f(u(r)), \qquad 0 < r < R. \tag{9}$$

Similarly, u_n is non-decreasing on $(0, R)$, for any $n \ge 1$.

In order to show that u is a large solution of (7), it remains to prove that $u(r) \to \infty$ as $r \nearrow R$. Assume the contrary. Then there exists $C > 0$ such that $u(r) < C$ for all $0 \le r < R$. Let $N_1 \ge 2C$ be fixed. The monotonicity of u_{N_1} and the fact that $u_{N_1}(r) \to N_1$ as $r \nearrow R$ imply the existence of some $r_1 \in (0, R)$ such that $C \le u_{N_1}(r)$, for $r \in [r_1, R)$. Hence

$$C \le u_{N_1}(r) \le u_{N_1+1}(r) \le \cdots \le u_n(r) \le u_{n+1}(r) \le \cdots \qquad \forall n \ge N_1, \quad \forall r \in [r_1, R).$$

Passing to the limit as $n \to \infty$, we obtain $u(r) \ge C$ for all $r \in [r_1, R)$, which is a contradiction.

Integrating (9) on $(0, r)$ and taking $r \nearrow R$ we find

$$\int_{u(0)}^{\infty} \frac{1}{f(t)}\,dt \le \frac{R^2}{2N}.$$

The conclusion of Lemma 1 is therefore proved. □

Proof of Theorem 1. By Theorem 5 in the Appendix, the boundary value problem

$$\begin{cases} \Delta v_n = p(x)f(v_n), & \text{in } \Omega \\ v_n = n, & \text{on } \partial\Omega \\ v_n \ge 0, \ v_n \not\equiv 0, & \text{in } \Omega \end{cases} \tag{10}$$

has a unique positive solution, for any $n \ge 1$.

We now claim that

(a) for all $x_0 \in \Omega$ there exist an open set $\mathcal{O} \subset\subset \Omega$ containing x_0 and $M_0 = M_0(x_0) > 0$ such that $v_n \le M_0$ in $\mathcal{O}$, for any $n \ge 1$;

(b) $\lim_{x\to\partial\Omega} v(x) = \infty$ where $v(x) = \lim_{n\to\infty} v_n(x)$.

We first remark that the sequence (v_n) is non-decreasing. Indeed, by Theorem 5 in the Appendix, the boundary value problem

$$\begin{cases} \Delta\zeta = \|p\|_\infty f(\zeta), & \text{in } \Omega \\ \zeta = 1, & \text{on } \partial\Omega \\ \zeta > 0, & \text{in } \Omega \end{cases}$$

has a unique solution. Then, by the maximum principle,

$$0 < \zeta \leq v_1 \leq \cdots \leq v_n \leq \cdots \qquad \text{in } \Omega\,. \tag{11}$$

We also observe that (a) and (b) are sufficient to conclude the proof. In fact, assertion (a) shows that the sequence (v_n) is uniformly bounded on every compact subset of Ω. Standard elliptic regularity arguments (see the proof of Lemma 3 in [9]) show that v is a solution of problem (1). Then, by (11) and (b), it follows that v is the desired solution.

To prove (a) we distinguish two cases :

CASE $p(x_0) > 0$. By the continuity of p, there exists a ball $B = B(x_0, r) \subset\subset \Omega$ such that

$$m_0 := \min\{p(x);\ x \in \overline{B}\} > 0.$$

Let w be a positive solution of problem

$$\begin{cases} \Delta w = m_0 f(w), & \text{in } B \\ w(x) \to \infty, & \text{as } x \to \partial B. \end{cases} \tag{12}$$

The existence of w follows by [7, Theorem III], due to Keller. By the maximum principle it follows that $v_n \leq w$ in B. Furthermore, w is bounded in $\overline{B(x_0, r/2)}$. Setting $M_0 = \sup\limits_{\mathcal{O}} w$, where $\mathcal{O} = B(x_0, r/2)$, we obtain (a).

CASE $p(x_0) = 0$. Our hypothesis $(p1)$ and the boundedness of Ω imply the existence of a domain $\mathcal{O} \subset\subset \Omega$ which contains x_0 such that $p > 0$ on $\partial\mathcal{O}$. The above case shows that for any $x \in \partial\mathcal{O}$ there exist a ball $B(x, r_x)$ strictly contained in Ω and a constant $M_x > 0$ such that $v_n \leq M_x$ on $B(x, r_x/2)$, for any $n \geq 1$. Since $\partial\mathcal{O}$ is compact, it follows that it may be covered by a finite number of such balls, say $B(x_i, r_{x_i}/2)$, $i = 1, \cdots, k_0$. Setting $M_0 = \max\{M_{x_1}, \cdots, M_{x_{k_0}}\}$ we have $v_n \leq M_0$ on $\partial\mathcal{O}$, for any $n \geq 1$. Applying the maximum principle we obtain $v_n \leq M_0$ in $\mathcal{O}$ and (a) follows.

Let us now consider the problem

$$\begin{cases} -\Delta z = p(x) & \text{in } \Omega \\ z = 0, & \text{on } \partial\Omega. \\ z \geq 0,\ z \not\equiv 0 & \text{in } \Omega \end{cases} \tag{13}$$

Applying Theorem 1 in Brezis-Oswald [2] we deduce that (13) has a unique solution which is positive in Ω, by the maximum principle.

We first observe that for proving (b) it is sufficient to show that

$$\int_{v(x)}^{\infty} \frac{dt}{f(t)} \leq z(x) \quad \text{for any } x \in \Omega. \tag{14}$$

By Lemma 1, the left hand-side of (14) is well defined in Ω. Fix $\varepsilon > 0$. Since $v_n = n$ on $\partial\Omega$, there is $n_1 = n_1(\varepsilon)$ such that

$$\int_{v_n(x)}^{\infty} \frac{dt}{f(t)} \leq \varepsilon(1+R^2)^{-1/2} \leq z(x) + \varepsilon(1+|x|^2)^{-1/2} \qquad \forall x \in \partial\Omega\,,\ \forall n \geq n_1\,, \tag{15}$$

where $R > 0$ is chosen so that $\overline{\Omega} \subset B(0,R)$.

In order to prove (14), it is enough to show that

$$\int_{v_n(x)}^{\infty} \frac{dt}{f(t)} \leq z(x) + \varepsilon(1+|x|^2)^{-1/2} \qquad \forall x \in \Omega\,,\ \forall n \geq n_1. \tag{16}$$

Indeed, putting $n \to \infty$ in (16) we deduce (14), since $\varepsilon > 0$ is arbitrarily chosen. Assume now, by contradiction, that (16) fails. Then

$$\max_{x \in \overline{\Omega}} \left\{ \int_{v_n(x)}^{\infty} \frac{dt}{f(t)} - z(x) - \varepsilon(1+|x|^2)^{-1/2} \right\} > 0.$$

Using (15) we see that the point where the maximum is achieved must lie in Ω. At this point, say x_0, we have

$$\begin{aligned}
&0 \geq \Delta \left(\int_{v_n(x)}^{\infty} \frac{dt}{f(t)} - z(x) - \varepsilon(1+|x|^2)^{-1/2} \right)_{|x=x_0} = \\
&\left(-\frac{1}{f(v_n)} \Delta v_n - \left(\frac{1}{f}\right)'(v_n) \cdot |\nabla v_n|^2 - \Delta z(x) - \varepsilon\Delta(1+|x|^2)^{-1/2} \right)_{|x=x_0} = \\
&\left(-p(x) - \left(\frac{1}{f}\right)'(v_n) \cdot |\nabla v_n|^2 + p(x) - \varepsilon\Delta(1+|x|^2)^{-1/2} \right)_{|x=x_0} = \\
&\left(-\left(\frac{1}{f}\right)'(v_n) \cdot |\nabla v_n|^2 + \varepsilon(N-3)(1+|x|^2)^{-3/2} + 3\,\varepsilon(1+|x|^2)^{-5/2} \right)_{|x=x_0} > 0.
\end{aligned}$$

This contradiction shows that inequality (16) holds and the proof of Theorem 1 is complete. □

3 Existence results for unbounded domains

In this section we are interested mainly in the question of finding and describing the behavior on the boundary and at infinity of the maximal solution to problem (1), where Ω is now an unbounded domain, possibly $\mathbf{R}^N$. For the significance of such a study we refer to Dynkin [5] where it is showed that there exist certain relations between hitting probabilities for superdiffusions and maximal solutions of (1) with $f(u) = u^\gamma$, $1 < \gamma \leq 2$.

It is clear that a unique normal large solution is necessarily a maximal solution. In view of this remark the problem of maximal solution seems to be connected with the uniqueness of large solutions. But this is not the best way to be followed because we lose the control if the uniqueness of large solutions fails. The advantage offered by our results is that we find a direct method which establishes an interesting connection between the maximal solution and *any* sequence of large solutions taken on bounded domains of the type given in condition $(p1)'$ in Ω.

Proof of Theorem 2. By Theorem 1, the boundary value problem

$$\begin{cases} \Delta v_n = p(x) f(v_n), & \text{in } \Omega_n \\ v_n(x) \to \infty\,, & \text{as } x \to \partial\Omega_n \\ v_n > 0\,, & \text{in } \Omega_n \end{cases} \tag{17}$$

has solution. Since $\overline{\Omega_n} \subset \Omega_{n+1}$ we can apply, for each $n \geq 1$, the maximum principle (in the same manner as in the uniqueness proof of Theorem 5 in the Appendix) in order to find that $v_n \geq v_{n+1}$ in Ω_n. Since $\mathbf{R}^N = \cup_{n=1}^{\infty} \Omega_n$ and $\overline{\Omega_n} \subset \Omega_{n+1}$ it follows that for every $x_0 \in \mathbf{R}^N$ there exists $n_0 = n_0(x_0)$ such that $x_0 \in \Omega_n$ for all $n \geq n_0$. In view of the monotonicity of the sequence $(v_n(x_0))_{n \geq n_0}$ we can define $U(x_0) = \lim_{n\to\infty} v_n(x_0)$. By applying the standard bootstrap argument (see [8, Theorem 1]) we find that $U \in C^{2,\alpha}_{\rm loc}(\mathbf{R}^N)$ and $\Delta U = p(x) f(U)$ in Ω.

We now prove that U is the maximal solution of problem (1). Indeed, let u be an arbitrary solution of (1). Applying again the maximum principle we obtain that $v_n \geq u$ in Ω_n for all $n \geq 1$. By the definition of U, it is clear that $U \geq u$ in $\mathbf{R}^N$.

We point out that U is independent of the choice of the sequence of domains Ω_n and the number of solutions of problem (17). This follows easily by the uniqueness of the maximal solution.

We suppose, in addition, that p satisfies $(p2)$ and we shall prove that U blows-up at infinity. For this aim, it is sufficient to find a positive function $w \in C(\mathbf{R}^N)$ such that $U \geq w$ in $\mathbf{R}^N$ and $w(x) \to \infty$ as $|x| \to \infty$. We first observe that $(p2)$ implies

$$K = \int_0^\infty r^{1-N} \left(\int_0^r \sigma^{N-1} \Phi(\sigma)\, d\sigma \right) dr < \infty. \tag{18}$$

Note that (18) is a simple consequence of the fact that for all $R > 0$ we have

$$\int_0^R r^{1-N}\left(\int_0^r \sigma^{N-1}\Phi(\sigma)\,d\sigma\right)dr = \frac{1}{2-N}\int_0^R \frac{d}{dr}(r^{2-N})\left(\int_0^r \sigma^{N-1}\Phi(\sigma)\,d\sigma\right)dr =$$
$$\frac{1}{2-N}R^{2-N}\int_0^R \sigma^{N-1}\Phi(\sigma)\,d\sigma - \frac{1}{2-N}\int_0^R r\Phi(r)\,dr \le \frac{1}{N-2}\int_0^\infty r\Phi(r)\,dr < \infty.$$

Using (18) and the maximum principle we obtain that the problem

$$\begin{cases} -\Delta z = \Phi(r), & r = |x| < \infty, \\ z(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

has a unique positive radial solution which is given by

$$z(r) = K - \int_0^r \sigma^{1-N}\left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\,d\tau\right)d\sigma, \qquad \forall r \ge 0.$$

Let w be the positive function defined implicitely by

$$z(x) = \int_{w(x)}^{\infty} \frac{dt}{f(t)}, \qquad \forall x \in \mathbf{R}^N. \tag{19}$$

Assumption $(f1)$ and L'Hospital rule yield

$$\lim_{t\searrow 0}\frac{f(t)}{t} = \lim_{t\searrow 0} f'(t) = f'(0) \in [0,\infty)\,,$$

which implies the existence of some $\delta > 0$ such that

$$\frac{f(t)}{t} < f'(0) + 1 \quad \text{for all } 0 < t < \delta.$$

Thus for every $s \in (0,\delta)$ we have

$$\int_s^\delta \frac{dt}{f(t)} > \frac{1}{f'(0)+1}\int_s^\delta \frac{dt}{t} = \frac{1}{f'(0)+1}(\ln\delta - \ln s).$$

It follows that $\lim_{s\searrow 0}\int_s^\delta \frac{dt}{f(t)} = \infty$, which gives the possibility to define w as in (19).

We claim that $w \le v_n$ in Ω_n for all $n \ge 1$. Obviously this inequality is true on $\partial\Omega_n$. Using the same arguments as in the proof of the inequality (26) in the Appendix (with Ω replaced by Ω_n) we obtain that for any $\varepsilon > 0$ and $n \ge 1$ we have

$$w(x) \le v_n(x) + \varepsilon(1+|x|^2)^{-1/2} \qquad \text{in } \Omega_n$$

and the claim follows. Consequently, $U \geq w$ in $\mathbf{R}^N$ and, by (19), $w(x) \to \infty$ as $|x| \to \infty$. This completes the proof. $\square$

Proof of Theorem 3. We argue in a similar manner as in the proof of Theorem 2, but with some changes due to the fact that $\Omega \neq \mathbf{R}^N$.

Let $(\Omega_n)_{n\geq 1}$ be the sequence of bounded smooth domains given by condition $(p1)'$. For $n \geq 1$ fixed, let v_n be a positive solution of problem (17) and recall that $v_n \geq v_{n+1}$ in Ω_n. Set $U(x) = \lim_{n\to\infty} v_n(x)$, for every $x \in \Omega$. With the same arguments as in Theorem 2 we find that U is a classical solution to (1) and that U is the maximal solution. Hence the first part of Theorem 3 is proved.

For the second part, in which $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$, we suppose that $(p2)$ is fulfilled, with $\Phi(r) = 0$ for $r \in [0, R]$. In order to prove that U is a normal large solution it is enough to show the existence of a positive function $w \in C(\mathbf{R}^N \setminus \overline{B(0,R)})$ such that $U \geq w$ in $\mathbf{R}^N \setminus \overline{B(0,R)}$, and $w(x) \to \infty$ as $|x| \to \infty$ and as $|x| \searrow R$. This will be done as in the proof of Theorem 2, with the function z given now as the unique positive radial solution of the problem

$$\begin{cases} -\Delta z = \Phi(r), & \text{if } |x| = r > R \\ z(x) \to 0 & \text{as } |x| \to \infty \\ z(x) \to 0 & \text{as } |x| \searrow R. \end{cases}$$

The uniqueness of z follows by the maximum principle. Moreover,

$$z(r) = \left(\frac{1}{R^{N-2}} - \frac{1}{r^{N-2}}\right) \int_R^\infty \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\, d\tau\right) d\sigma - \frac{1}{R^{N-2}} \int_R^r \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\, d\tau\right) d\sigma.$$

This completes the proof. $\square$

4 Proof of Theorem 4

Let u be an entire large solution of problem (1). Define

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \left(\int_a^{u(x)} \frac{dt}{f(t)}\right) dS = \frac{1}{\omega_N} \int_{|\xi|=1} \left(\int_a^{u(r\xi)} \frac{dt}{f(t)}\right) dS,$$

where ω_N denotes the surface area of the unit sphere in $\mathbf{R}^N$ and a is chosen such that $a \in (0, u_0)$, where $u_0 = \inf_{\mathbf{R}^N} u > 0$. By the divergence theorem we have

$$\bar{u}'(r) = \frac{1}{\omega_N} \int\limits_{|\xi|=1} \frac{1}{f(u(r\xi))} \nabla u(r\xi) \cdot \xi \, dS = \frac{1}{\omega_N r^N} \int\limits_{|y|=r} \frac{1}{f(u(y))} \nabla u(y) \cdot y \, dS =$$
$$\frac{1}{\omega_N r^N} \int\limits_{|y|=r} \nabla \left(\int\limits_a^{u(y)} \frac{dt}{f(t)} \right) \cdot y \, dS = \frac{1}{\omega_N r^{N-1}} \int\limits_{|y|=r} \frac{\partial}{\partial \nu} \left(\int\limits_a^{u(y)} \frac{dt}{f(t)} \right) dS =$$
$$\frac{1}{\omega_N r^{N-1}} \int\limits_{B(0,r)} \Delta \left(\int\limits_a^{u(x)} \frac{dt}{f(t)} \right) dx.$$

Since u is a positive classical solution it follows that

$$|\bar{u}'(r)| \leq Cr \to 0 \qquad \text{as } r \to 0\,.$$

On the other hand

$$\omega_N \left(R^{N-1} \bar{u}'(R) - r^{N-1} \bar{u}'(r) \right) = \int\limits_D \Delta \left(\int\limits_a^{u(x)} \frac{1}{f(t)} \, dt \right) dx = \int\limits_r^R \left(\int\limits_{|x|=z} \Delta \left(\int\limits_a^{u(x)} \frac{dt}{f(t)} \right) dS \right) dz,$$

where $D = \{x \in \mathbf{R}^N : r < |x| < R\}$. Dividing by $R - r$ and letting $R \to r$ we find

$$\omega_N (r^{N-1} \bar{u}'(r))' = \int\limits_{|x|=r} \Delta \left(\int\limits_a^{u(x)} \frac{dt}{f(t)} \right) dS = \int\limits_{|x|=r} \operatorname{div} \left(\frac{1}{f(u(x))} \nabla u(x) \right) dS =$$
$$\int\limits_{|x|=r} \left[\left(\frac{1}{f} \right)' (u(x)) \cdot |\nabla u(x)|^2 + \frac{1}{f(u(x))} \Delta u(x) \right] dS \leq \int\limits_{|x|=r} \frac{p(x) f(u(x))}{f(u(x))} \, dS \leq \omega_N r^{N-1} \Phi(r).$$

Integrating the above inequality yields

$$\bar{u}(r) \leq \bar{u}(0) + \int\limits_0^r \sigma^{1-N} \left(\int\limits_0^\sigma \tau^{N-1} \Phi(\tau) \, d\tau \right) d\sigma \qquad \forall r \geq 0. \tag{20}$$

Since $(p2)$ implies (18) we have

$$\bar{u}(r) \leq \bar{u}(0) + K \qquad \forall r \geq 0.$$

Thus $\bar{u}$ is bounded and assuming that (4) is not fulfilled it follows that u cannot be a large solution. $\square$

5 Appendix

The following result is mentioned without proof in Marcus [12] and it was applied several times in this paper. For the sake of completeness we present in this section a simple proof of this theorem.

Theorem 5 *Let Ω be a bounded domain. Assume that $p \in C^{0,\alpha}(\overline{\Omega})$ is a non-negative function, f satisfies $(f1)$ and $g : \partial\Omega \to (0, \infty)$ is continuous. Then the boundary value problem*

$$\begin{cases} \Delta u = p(x) f(u), & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \\ u \geq 0, \ u \not\equiv 0, & \text{in } \Omega \end{cases} \tag{21}$$

has a unique classical solution, which is positive.

Proof of Theorem 5. We first observe that the function $u^+(x) = n$ is a super-solution of problem (21), provided that n is sufficiently large. To find a positive sub-solution, we look for an arbitrary positive solution to the following auxiliary problem

$$\Delta v = \Phi(r) \qquad \text{in } A(\underline{r}, \overline{r}) = \{x \in \mathbf{R}^N;\ \underline{r} < |x| < \overline{r}\} \tag{22}$$

where

$$\underline{r} = \inf \{\tau > 0;\ \partial B(0,\tau) \cap \overline{\Omega} \neq \emptyset\}, \quad \overline{r} = \sup \{\tau > 0;\ \partial B(0,\tau) \cap \overline{\Omega} \neq \emptyset\}$$

$$\Phi(r) = \max_{|x|=r} p(x) \quad \text{for any } r \in [\underline{r}, \overline{r}].$$

The function

$$v(r) = 1 + \int_{\underline{r}}^{r} \sigma^{1-N} \left(\int_0^{\sigma} \tau^{N-1} \Phi(\tau)\, d\tau \right) d\sigma, \qquad \underline{r} \leq r \leq \overline{r}$$

verifies equation (22). The assumptions on f and g imply

$$g_0 := \min_{\partial\Omega} g > 0 \quad \text{and} \quad \lim_{z \searrow 0} \int_z^{g_0} \frac{dt}{f(t)} = \infty.$$

This will be used to justify the existence of a positive number c such that

$$\max_{\partial\Omega} v = \int_c^{g_0} \frac{dt}{f(t)}. \tag{23}$$

Next, we define the function u_- such that

$$v(x) = \int_c^{u_-(x)} \frac{dt}{f(t)}, \qquad \forall x \in \Omega. \tag{24}$$

It turns out that u_- is a positive sub-solution of problem (21). Indeed, it is clear that

$$u_- \in C^2(\Omega) \cap C(\overline{\Omega}) \quad \text{and} \quad u_- \geq c \text{ in } \Omega.$$

On the hand, from (22), (24) and $(f1)$ it follows that

$$p(x) \leq \Delta v(x) = \frac{1}{f(u_-(x))}\Delta u_-(x) + \left(\frac{1}{f}\right)'(u_-(x)) \cdot |\nabla u_-(x)|^2 \leq \frac{1}{f(u_-(x))}\Delta u_-(x) \qquad \text{in } \Omega,$$

which yields

$$\Delta u_-(x) \geq p(x) f(u_-(x)) \qquad \text{in } \Omega.$$

On the other hand, taking into account (23) and (24) we find

$$u_-(x) \leq g(x) \qquad \forall x \in \partial\Omega.$$

So, we have proved that u_- is a positive sub-solution to problem (21). Therefore this problem has at least a positive solution u. Furthermore, taking into account the regularity of p and f, a standard boot-strap argument based on Schauder and Hölder regularity shows that $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Let us now assume that u_1 and u_2 are arbitrary solutions of (21). In order to prove the uniqueness, it is enough to show that $u_1 \geq u_2$ in Ω. Denote

$$\omega := \{x \in \Omega;\ u_1(x) < u_2(x)\}$$

and suppose that $\omega \neq \emptyset$. Then the function $\tilde{u} = u_1 - u_2$ satisfies

$$\begin{cases} \Delta \tilde{u} = p(x)(f(u_1) - f(u_2)), & \text{in } \omega \\ \tilde{u} = 0, & \text{on } \partial\omega\,. \end{cases} \tag{25}$$

Since f is non-decreasing and $p \geq 0$, it follows by (25) that $\tilde{u}$ is a super-harmonic function in ω which vanishes on $\partial\omega$. Thus, by the maximum principle, either $\tilde{u} \equiv 0$ or $\tilde{u} > 0$ in ω, which yield a contradiction. Thus $u_1 \geq u_2$ in Ω.

We give in what follows an alternative proof for the uniqueness. Let u_1, u_2 be two arbitrary solutions of problem (21). As above, it is enough to show that $u_1 \geq u_2$ in Ω. Fix $\varepsilon > 0$. We claim that

$$u_2(x) \leq u_1(x) + \varepsilon(1 + |x|^2)^{-1/2} \qquad \text{for any } x \in \Omega. \tag{26}$$

Suppose the contrary. Since (26) is obviously fulfilled on $\partial\Omega$, we deduce that

$$\max_{x \in \overline{\Omega}} \left\{u_2(x) - u_1(x) - \varepsilon(1 + |x|^2)^{-1/2}\right\}$$

is achieved in Ω. At that point we have

$$0 \geq \Delta\left(u_2(x) - u_1(x) - \varepsilon(1+|x|^2)^{-1/2}\right) = p(x)\left(f(u_2(x)) - f(u_1(x))\right) - \varepsilon\Delta(1+|x|^2)^{-1/2} =$$
$$p(x)\left(f(u_2(x)) - f(u_1(x))\right) + \varepsilon(N-3)(1+|x|^2)^{-3/2} + 3\,\varepsilon(1+|x|^2)^{-5/2} > 0\,,$$

which is a contradiction. Since $\varepsilon > 0$ is chosen arbitrarily, inequality (26) implies $u_2 \leq u_1$ in Ω. □

We point out that the hypothesis that f is differentiable in the origin is essential in order to find a *positive* solution to problem (21). Indeed, consider $\Omega = B_1$, and $f(u) = u^{(\beta-2)/\beta}$, where $\beta > 2$. Choose $p \equiv 1$ and $g \equiv C$ on ∂B_1, where $C = \left(\beta^2 + (N-2)\beta\right)^{-\beta/2}$. For this choice of Ω, p, f and g, the function $u(r) = Cr^\beta$, $0 \leq r \leq 1$, is the unique solution of problem (21), but $u(0) = 0$.

Under the hypotheses on f made in the statement of Theorem 5, except f is of class C^1 at the origin (but $f \in C^{0,\alpha}$ in $u = 0$), problem (21) has a unique solution which may vanish in Ω. For this purpose it is sufficient to choose as a sub-solution in the above proof the function $u_- = 0$.

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Existence and uniqueness of blow-up solutions for a class of logistic equations*

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This paper is dedicated with esteem to Professor Viorel Barbu on his 60th birthday

Abstract. Let f be a non-negative C^1-function on $[0,\infty)$ such that $f(u)/u$ is increasing and $\int_1^\infty 1/\sqrt{F(t)}\,dt<\infty$, where $F(t)=\int_0^t f(s)\,ds$. Assume $\Omega\subset\mathbf{R}^N$ is a smooth bounded domain, a is a real parameter and $b\geq 0$ is a continuous function on $\overline{\Omega}$, $b\not\equiv 0$. We consider the problem $\Delta u+au=b(x)f(u)$ in Ω and we prove a necessary and sufficient condition for the existence of positive solutions that blow-up at the boundary. We also deduce several existence and uniqueness results for a related problem, subject to homogeneous Dirichlet, Neumann or Robin boundary condition.

1 Introduction and the main results

Consider the semilinear elliptic equation

$$\Delta u+au=b(x)f(u)\qquad \text{in } \Omega, \tag{1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$, $N\geq 3$. Let a be a real parameter and $b\in C^{0,\mu}(\overline{\Omega})$, $0<\mu<1$, such that $b\geq 0$ and $b\not\equiv 0$ in Ω. Set

$$\Omega_0=\operatorname{int}\{x\in\Omega:\ b(x)=0\}$$

and suppose, throughout, that $\overline{\Omega}_0\subset\Omega$ and $b>0$ on $\Omega\setminus\overline{\Omega}_0$. Assume that $f\in C^1[0,\infty)$ satisfies

$(A_1)\quad f\geq 0$ and $f(u)/u$ is increasing on $(0,\infty)$.

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Following Alama and Tarantello [1], define by H_∞ the Dirichlet Laplacian on Ω_0 as the unique self-adjoint operator associated to the quadratic form $\psi(u) = \int_\Omega |\nabla u|^2\,dx$ with form domain

$$H^1_D(\Omega_0) = \{u \in H^1_0(\Omega) :\ u(x) = 0 \quad \text{for a.e. } x \in \Omega \setminus \Omega_0\}.$$

If $\partial\Omega_0$ satisfies the exterior cone condition then, according to [1], $H^1_D(\Omega_0)$ coincides with $H^1_0(\Omega_0)$ and H_∞ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of H_∞ in Ω_0. We understand $\lambda_{\infty,1} = \infty$ if $\Omega_0 = \emptyset$.

Set $\mu_0 := \lim_{u\searrow 0} \frac{f(u)}{u}$, $\mu_\infty := \lim_{u\to\infty} \frac{f(u)}{u}$, and denote by $\lambda_1(\mu_0)$ (resp., $\lambda_1(\mu_\infty)$) the first eigenvalue of the operator $H_{\mu_0} = -\Delta + \mu_0 b$ (resp., $H_{\mu_\infty} = -\Delta + \mu_\infty b$) in $H^1_0(\Omega)$. Recall that $\lambda_1(+\infty) = \lambda_{\infty,1}$.

Alama and Tarantello [1] proved that problem (1) subject to the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

has a positive solution u_a if and only if $a \in (\lambda_1(\mu_0), \lambda_1(\mu_\infty))$. Moreover, u_a is the unique positive solution for (1)+(2) (see [1, Theorem A (bis)]). We shall refer to the combination of (1)+(2) as problem (E_a).

Our first aim is to give a corresponding necessary and sufficient condition, but for the existence of *large* (or *explosive*) solutions of (1). A solution u of (1) such that $u \geq 0$ in Ω and $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ will be called a large solution. Cf. Corollary 5 in the Appendix, if such a solution exists, then it is *positive* even if f satisfies a weaker condition than (A_1), namely

$(A_1)'$ $\quad f(0) = 0$, $f' \geq 0$ and $f > 0$ on $(0, \infty)$.

Problems related to large solutions have a long history and are studied by many authors and in many contexts. Singular value problems of this type go back to the pioneering work [29] on the equation $\Delta u = e^u$ in the space, and were later studied under the general form $\Delta u = f(u)$ in N–dimensional domains. We refer only to [3]–[6], [11], [15], [16], [21], [22], [24]–[26], and [31]. We also point out the paper [30], where there are studied large solutions of the problem

$$\Delta u = K(x) u^{(N+2)/(N-2)}$$

in a ball, in particular for questions of existence, uniqueness and boundary behaviour.

Keller [20] and Osserman [27] supplied a necessary and sufficient condition on f for the existence of large solutions to (1) when $a \equiv 0$, $b \equiv 1$ and f is assumed to fulfill $(A_1)'$. More precisely, f must satisfy the Keller-Osserman condition (see [20, 27]),

$$(A_2) \qquad \int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty\,, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$$

Keeping this in mind and using Theorem 4 in the Appendix we find that our problem (1) can have large solutions only if the Keller-Osserman condition (A_2) is fulfilled (see Remark 2 in Section 3). Furthermore, when this really happens, our first result gives the maximal interval for the parameter a that ensures the existence of large solutions to problem (1). More precisely, we prove

Theorem 1 *Assume that f satisfies conditions (A_1) and (A_2). Then problem (1) has a large solution if and only if $a \in (-\infty, \lambda_{\infty,1})$.*

We point out that our framework in the above result includes the case when b vanishes at some points on $\partial\Omega$, or even if $b \equiv 0$ on $\partial\Omega$. In this sense, our result responds to a question raised to one of us by Professor Haim Brezis in Paris, May 2001.

Denote by $\mathcal{D}$ and $\mathcal{R}$ the boundary operators

$$\mathcal{D}u := u \qquad \text{and} \qquad \mathcal{R}u := \partial_\nu u + \beta(x)u,$$

where ν is the unit outward normal to $\partial\Omega$, and $\beta \in C^{1,\mu}(\partial\Omega)$ is non-negative. Hence, $\mathcal{D}$ is the *Dirichlet* boundary operator and $\mathcal{R}$ is either the *Neumann* boundary operator, if $\beta \equiv 0$, or the *Robin* boundary operator, if $\beta \not\equiv 0$. Throughout this work, $\mathcal{B}$ can define any of these boundary operators.

Note that the Robin condition $\mathcal{R} = 0$ relies essentially to heat flow problems in a body with constant temperature in the surrounding medium. More generally, if α and β are smooth functions on $\partial\Omega$ such that $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, then the boundary condition $Bu = \alpha\partial_\nu u + \beta u = 0$ represents the exchange of heat at the surface of the reactant by Newtonian cooling. Moreover, the boundary condition $Bu = 0$ is called isothermal (Dirichlet) condition if $\alpha \equiv 0$, and it becomes an adiabatic (Neumann) condition if $\beta \equiv 0$. An intuitive meaning of the condition $\alpha + \beta > 0$ on $\partial\Omega$ is that, for the diffusion process described by problem (1), either the reflection phenomenon or the absorption phenomenon may occur at each point of the boundary.

If $f(u) = u^p$ $(p > 1)$, the semilinear elliptic problem

$$\begin{cases} \Delta u + au = b(x)u^p & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

is basic population model (see, e.g., [18]) and is also related to some prescribed curvature problems in Riemannian geometry (see, e.g., [28] and [19]). The existence of positive solutions of (3) has been intensively studied; see for example [1], [2], [12], [13], [17] and [28].

If b is positive on $\overline{\Omega}$ then (3) is known as the logistic equation and it has a unique positive solution if and only if $a > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases}$$

We are now concerned with the following boundary blow-up problem

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega \setminus \overline{\Omega}_0\,, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega\,, \\ u = \infty & \text{on } \partial\Omega_0\,, \end{cases} \tag{4}$$

where $b > 0$ on $\partial\Omega$, while $\overline{\Omega}_0$ is non-empty, connected and with smooth boundary. Here, $u = \infty$ on $\partial\Omega_0$ means that $u(x) \to \infty$ as $x \in \Omega \setminus \overline{\Omega}_0$ and $d(x) := \text{dist}\,(x, \Omega_0) \to 0$.

The question of existence and uniqueness of positive solutions for problem (4) in the case of pure superlinear power in the non-linearity is treated by Du-Huang [16]. Our next results extend their previous paper to the case of much more general non-linearities of Keller-Osserman type.

In the following, by $(\tilde{A}_1)$ we mean that (A_1) is fulfilled and there exists $\lim_{u\to\infty}(F/f)'(u) := \gamma$. Then, $\gamma \geq 0$. Moreover, $\gamma \leq 1/2$ if, in addition, (A_2) is satisfied (see Lemma 6).

We prove

Theorem 2 *Let $(\tilde{A}_1)$ and (A_2) hold. Then, for any $a \in \mathbf{R}$, problem (4) has a minimal (resp., maximal) positive solution $\underline{U}_a$ (resp., $\overline{U}_a$).*

In proving Theorem 2 we rely on an appropriate comparison principle (see Lemma 3) which allows us to prove that $(u_n)_{n\geq 1}$ is non-decreasing, where u_n is the unique positive solution of problem (31) (in Lemma 9) with $\Phi \equiv n$. The minimal positive solution of (4) will be obtained as the limit of the sequence $(u_n)_{n\geq 1}$. Note that, since $b = 0$ on $\partial\Omega_0$, the main difficulty is related to the construction of an upper bound of this sequence (see Lemma 10) which must fit to our general framework. To overcome it, we find an equivalent criterion to the Keller-Osserman condition (A_2) (see Lemma 7). Next, we deduce the maximal positive solution of (4) as the limit of the non-increasing sequence $(v_m)_{m\geq m_1}$ provided m_1 is large so that $\Omega_{m_1} \subset\subset \Omega$. We denoted by v_m the minimal positive solution of (4) with Ω_0 replaced by

$$\Omega_m := \{x \in \Omega : \ d(x) < 1/m\}, \qquad m \geq m_1. \tag{5}$$

The next question is whether one can conclude the uniqueness of positive solutions of problem (4). We recall first what is already known in this direction. When $f(u) = u^p$, $p > 1$, Du-Huang [16] proved the uniqueness of solution to problem (4) and established its behavior near $\partial\Omega_0$, under the assumption

$$\lim_{d(x)\searrow 0} \frac{b(x)}{[d(x)]^\tau} = c \quad \text{for some positive constants } \tau, c > 0. \tag{6}$$

We shall give a general uniqueness result provided that b and f satisfy the following assumptions:

(B_1) $\displaystyle\lim_{d(x)\searrow 0} \frac{b(x)}{k(d(x))} = c$ for some constant $c > 0$, where $0 < k \in C^1(0, \delta_0)$ is increasing and satisfies

(B_2) $\displaystyle K(t) = \frac{\int_0^t \sqrt{k(s)}\,ds}{\sqrt{k(t)}} \in C^1[0, \delta_0)$, for some $\delta_0 > 0$.

Assume there exist $\zeta > 0$ and $t_0 \geq 1$ such that

(A_3) $f(\xi t) \leq \xi^{1+\zeta} f(t), \quad \forall \xi \in (0,1), \quad \forall t \geq t_0/\xi$

(A_4) the mapping $(0,1] \ni \xi \longmapsto A(\xi) = \lim_{u\to\infty} \dfrac{f(\xi u)}{\xi f(u)}$ is a continuous positive function.

Our uniqueness result is

Theorem 3 *Assume the conditions $(\tilde{A}_1)$ with $\gamma \neq 0$, (A_3), (A_4), (B_1) and (B_2) hold. Then, for any $a \in \mathbf{R}$, problem (4) has a unique positive solution U_a. Moreover,*

$$\lim_{d(x)\searrow 0} \frac{U_a(x)}{h(d(x))} = \xi_0,$$

where h is defined by

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t \sqrt{k(s)}\,ds, \quad \forall t \in (0, \delta_0) \tag{7}$$

and ξ_0 is the unique positive solution of $A(\xi) = \dfrac{K'(0)(1-2\gamma) + 2\gamma}{c}$.

Remark 1 *(a)* $(A_1)+(A_3) \Rightarrow (A_2)$. Indeed, $\lim_{u\to\infty} \frac{f(u)}{u^{1+\zeta}} > 0$ since $\frac{f(t)}{t^{1+\zeta}}$ is non-decreasing for $t \geq t_0$.
(b) $K'(0)(1-2\gamma)+2\gamma \in (0,1]$ *when* $(\tilde{A}_1)$ *with* $\gamma \neq 0$, (A_2), (B_1) *and* (B_2) *hold (see Lemma 8).*
(c) *The function* $(0,\infty) \ni \xi \longmapsto A(\xi) \in (0,\infty)$ *is bijective when* (A_3) *and* (A_4) *hold (see Lemma 12).*

Among the non-linearities f that satisfy the assumptions of Theorem 3 we note: (i) $f(u)=u^p$, $p>1$; (ii) $f(u)=u^p\ln(u+1)$, $p>1$; (iii) $f(u)=u^p\arctan u$, $p>1$.

Theorem 2.8 in [16] follows by applying Theorem 3 with $f(u)=u^p$, $p>1$ and $k(t)=t^\tau$ for $t>0$. However, our result proves the uniqueness for a larger class of functions b than in [16]. Indeed, if (B_1) is satisfied with $k(t)=e^{-1/t}$ for $t>0$, then the uniqueness remains despite of (6) which is not valid.

The above results also apply to problems on Riemannian manifolds if Δ is replaced by the Laplace–Beltrami operator

$$\Delta_B = \frac{1}{\sqrt{c}}\frac{\partial}{\partial x_i}\left(\sqrt{c}\, a_{ij}(x)\frac{\partial}{\partial x_i}\right), \qquad c := \det\left(a_{ij}\right),$$

with respect to the metric $ds^2 = c_{ij}\,dx_i dx_j$, where (c_{ij}) is the inverse of (a_{ij}). In this case our results apply to concrete problems arising in Riemannian geometry. For instance, (cf. Loewner-Nirenberg [24] and Li [23]) if Ω is replaced by the standard N–sphere (S^N, g_0), Δ is the Laplace-Beltrami operator Δ_{g_0}, $a = N(N-2)/4$, and $f(u) = (N-2)/[4(N-1)]\,u^{(N+2)/(N-2)}$, we find the prescribing scalar curvature equation on S^N.

2 Comparison principles

Throughout this section, we assume that f is continuous on $(0,\infty)$ and $\frac{f(u)}{u}$ is increasing on $(0,\infty)$.

Lemma 1 *Assume* ω *is a bounded domain and* $p \in C^{0,\mu}(\overline{\omega})$ *is a positive function in* ω.
If u_1, $u_2 \in C^2(\omega)$ *are positive functions in* ω *and*

$$\Delta u_1 + au_1 - p(x)f(u_1) \leq 0 \leq \Delta u_2 + au_2 - p(x)f(u_2) \quad \text{in } \omega \tag{8}$$

$$\limsup_{\mathrm{dist}\,(x,\partial\omega)\to 0}\ (u_2-u_1)(x) \leq 0 \tag{9}$$

then $u_1 \geq u_2$ *in* ω.

Proof. We use the same method as in the proof of Lemma 1.1 in Marcus-Veron [26] (see also [16, Lemma 2.1]), that goes back to Benguria-Brezis-Lieb [7].

By (8) we obtain, for any non-negative function $\varphi \in H^1(\omega)$ with compact support in ω,

$$\int_\omega \left(\nabla u_1 \cdot \nabla\varphi - au_1\varphi + p(x)f(u_1)\varphi\right)\,dx \geq 0 \geq \int_\omega \left(\nabla u_2 \cdot \nabla\varphi - au_2\varphi + p(x)f(u_2)\varphi\right)\,dx\,. \tag{10}$$

Let $\varepsilon_1 > \varepsilon_2 > 0$ and denote

$$\omega_+(\varepsilon_1,\varepsilon_2) = \{x \in \omega:\ u_2(x)+\varepsilon_2 > u_1(x)+\varepsilon_1\}.$$

$$v_i = (u_i+\varepsilon_i)^{-1}\left((u_2+\varepsilon_2)^2-(u_1+\varepsilon_1)^2\right)^+, \quad i=1,2.$$

Notice that $v_i \in H^1_{\text{loc}}(\omega)$ and, in view of (9), it has compact support in ω. Using (10) with $\varphi = v_i$ and taking into account the fact that v_i vanishes outside $\omega_+(\varepsilon_1, \varepsilon_2)$ we find

$$\begin{aligned} &-\int_{\omega_+(\varepsilon_1,\varepsilon_2)} (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1)\, dx \\ &\quad \geq \int_{\omega_+(\varepsilon_1,\varepsilon_2)} p(x)(f(u_2)v_2 - f(u_1)v_1)\, dx + a \int_{\omega_+(\varepsilon_1,\varepsilon_2)} (u_1 v_1 - u_2 v_2)\, dx. \end{aligned} \tag{11}$$

A simple computation shows that the integral in the left-hand side of (11) equals

$$-\int_{\omega_+(\varepsilon_1,\varepsilon_2)} \left(\left| \nabla u_2 - \frac{u_2+\varepsilon_2}{u_1+\varepsilon_1} \nabla u_1 \right|^2 + \left| \nabla u_1 - \frac{u_1+\varepsilon_1}{u_2+\varepsilon_2} \nabla u_2 \right|^2 \right) dx \leq 0\,.$$

Passing to the limit as $0 < \varepsilon_2 < \varepsilon_1 \to 0$, the first term in the right hand-side of (11) converges to

$$\int_{\omega_+(0,0)} p(x) \left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \right) (u_2^2 - u_1^2)\, dx\,,$$

while the other term converges to 0. Hence, we avoid a contradiction only in the case that $\omega_+(0,0)$ has measure 0, which means that $u_1 \geq u_2$ on ω. ■

With the same arguments Lemma 1 can be written in the following more general form.

Lemma 2 *Let ω be a bounded domain. Assume that p, q, r are $C^{0,\mu}$-functions on $\overline{\omega}$ such that $r \geq 0$ and $p > 0$ in ω. If u_1, $u_2 \in C^2(\omega)$ are positive functions in ω and*

$$\Delta u_1 + q(x)u_1 - p(x)f(u_1) + r(x) \leq 0 \leq \Delta u_2 + q(x)u_2 - p(x)f(u_2) + r(x) \quad \text{in } \omega \tag{12}$$

$$\limsup_{\text{dist}\,(x,\partial\omega)\to 0} \; (u_2 - u_1)(x) \leq 0 \tag{13}$$

then $u_1 \geq u_2$ in ω.

The next result extends Lemma 2.1 in Du-Huang [16].

Lemma 3 *Assume $\omega \subset\subset \Omega$ and $p \in C^{0,\mu}(\overline{\Omega} \setminus \omega)$ is a positive function in $\Omega \setminus \overline{\omega}$.*

If u_1, $u_2 \in C^2(\overline{\Omega} \setminus \overline{\omega})$ are positive functions in $\Omega \setminus \overline{\omega}$ and

$$\Delta u_1 + a u_1 - p(x)f(u_1) \leq 0 \leq \Delta u_2 + a u_2 - p(x)f(u_2) \quad \text{in } \Omega \setminus \overline{\omega} \tag{14}$$

$$\mathcal{B}u_1 \geq 0 \geq \mathcal{B}u_2 \quad \text{on } \partial\Omega\,; \qquad \limsup_{\text{dist}\,(x,\partial\omega)\to 0} \; (u_2 - u_1)(x) \leq 0\,, \tag{15}$$

then $u_1 \geq u_2$ on $\overline{\Omega} \setminus \overline{\omega}$.

Proof. We distinguish 2 cases:

CASE 1: $\mathcal{B} = \mathcal{D}$. The assertion is an easy consequence of Lemma 1.

CASE 2: $\mathcal{B} = \mathcal{R}$. Let φ_1, φ_2 be two non-negative C^2-functions on $\overline{\Omega} \setminus \omega$ vanishing near $\partial\omega$. Multiplying in (14) the first inequality (resp., the second one) by φ_1 (resp., φ_2) and applying integration by parts together with (15) we deduce that

$$\begin{aligned} &-\int_{\tilde{\Omega}} (\nabla u_2 \cdot \nabla \varphi_2 - \nabla u_1 \cdot \nabla \varphi_1)\, dx - \int_{\partial\Omega} \beta(x)(u_2\varphi_2 - u_1\varphi_1)\, dS(x) \\ &\quad \geq \int_{\tilde{\Omega}} p(x)(f(u_2)\varphi_2 - f(u_1)\varphi_1)\, dx + a \int_{\tilde{\Omega}} (u_1\varphi_1 - u_2\varphi_2)\, dx, \end{aligned} \tag{16}$$

where $\tilde{\Omega} := \Omega \setminus \overline{\omega}$. Let $\varepsilon_1 > \varepsilon_2 > 0$ and denote

$$\Omega_+(\varepsilon_1, \varepsilon_2) = \{x \in \tilde{\Omega} : \ u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1\}.$$
$$v_i = (u_i + \varepsilon_i)^{-1} \left((u_2 + \varepsilon_2)^2 - (u_1 + \varepsilon_1)^2\right)^+, \quad i = 1, 2.$$

Since v_i can be approximated closely in the $H^1 \cap L^\infty$-topology on $\overline{\Omega} \setminus \omega$ by non-negative C^2-functions vanishing near $\partial\omega$, it follows that (16) holds for v_i taking place of φ_i. Since v_i vanishes outside the set $\Omega_+(\varepsilon_1, \varepsilon_2)$ relation (16) becomes

$$\begin{aligned} & -\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1)\, dx - \int_{\partial\Omega} \beta(x)(u_2 v_2 - u_1 v_1)\, dS(x) \\ & \quad \geq \int_{\Omega_+(\varepsilon_1,\varepsilon_2)} p(x)(f(u_2)v_2 - f(u_1)v_1)\, dx + a \int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (u_1 v_1 - u_2 v_2)\, dx. \end{aligned} \tag{17}$$

As $\varepsilon_1 \to 0$ (recall that $\varepsilon_1 > \varepsilon_2 > 0$) the second term on the left hand-side of (17) converges to 0. From now on, the course of the proof is the same as in Lemma 1. This completes the proof. ∎

3 Large solutions of Problem (1)

Remark 2 *Assuming (A_1), problem (1) can have large solutions only if f satisfies the Keller-Osserman condition (A_2).*

Suppose, a priori, that problem (1) has a large solution u_∞. Set $\tilde{f}(u) = |a|u + \|b\|_\infty f(u)$ for $u \geq 0$. Notice that $\tilde{f} \in C^1[0, \infty)$ satisfies $(A_1)'$. For any $n \geq 1$, consider the problem

$$\begin{cases} \Delta u = \tilde{f}(u) & \text{in } \Omega\,, \\ u = n & \text{on } \partial\Omega\,, \\ u \geq 0 & \text{in } \Omega\,. \end{cases}$$

By Theorem 4 in the Appendix, this problem has a unique solution, say u_n, which, moreover, is positive in $\overline{\Omega}$. Applying Lemma 2 for $q \equiv -|a|$, $p \equiv \|b\|_\infty$, $r \equiv 0$ and $\omega = \Omega$ we obtain

$$0 < u_n \leq u_{n+1} \leq u_\infty \quad \text{in } \Omega, \quad \forall n \geq 1.$$

Thus, for every $x \in \Omega$, we can define $\bar{u}(x) = \lim_{n\to\infty} u_n(x)$. Moreover, since (u_n) is uniformly bounded on every compact subset of Ω, standard elliptic regularity arguments show that $\bar{u}$ is a positive large solution of the problem $\Delta u = \tilde{f}(u)$. It follows that $\tilde{f}$ satisfies the Keller-Osserman condition (A_2). Then, by (A_1), $\mu_\infty := \lim_{u\to\infty} f(u)/u > 0$ which yields $\lim_{u\to\infty} \tilde{f}(u)/f(u) = |a|/\mu_\infty + \|b\|_\infty < \infty$. Consequently, our claim follows.

Typical examples of non-linearities satisfying (A_1) and (A_2) are:

(i) $f(u) = e^u - 1$; (ii) $f(u) = u^p$, $p > 1$; (iii) $f(u) = u[\ln(u+1)]^p$, $p > 2$.

Remark 3 *We have $\mu_\infty := \lim_{u\to\infty} f(u)/u = \lim_{u\to\infty} f'(u) = \infty$.*

Indeed, by l'Hospital's rule, $\lim_{u\to\infty} F(u)/u^2 = \mu_\infty/2$. But, by (A_2), we deduce that $\mu_\infty = \infty$. Then, by (A_1) we find that $f'(u) \geq f(u)/u$ for any $u > 0$, which shows that $\lim_{u\to\infty} f'(u) = \infty$.

Proof of Theorem 1

A. NECESSARY CONDITION. Let u_∞ be a large solution of problem (1). Corollary 5 in the Appendix implies that u_∞ is positive. Suppose $\lambda_{\infty,1}$ is finite. Arguing by contradiction, let us assume $a \geq \lambda_{\infty,1}$. Set $\lambda \in (\lambda_1(\mu_0), \lambda_{\infty,1})$ and denote by u_λ the unique positive solution of problem (E_a) with $a = \lambda$. We have

$$\begin{cases} \Delta(Mu_\infty) + \lambda_{\infty,1}(Mu_\infty) \leq b(x)f(Mu_\infty) & \text{in } \Omega\,, \\ Mu_\infty = \infty & \text{on } \partial\Omega\,, \\ Mu_\infty \geq u_\lambda & \text{in } \Omega\,, \end{cases}$$

where $M := \max\{\max_{\overline{\Omega}} u_\lambda / \min_\Omega u_\infty;\, 1\}$. By the sub-super solution method we conclude that problem (E_a) with $a = \lambda_{\infty,1}$ has at least a positive solution (between u_λ and Mu_∞). But this is a contradiction. So, necessarily, $a \in (-\infty, \lambda_{\infty,1})$.

B. SUFFICIENT CONDITION. This will be proved with the aid of several results. We assume, until the end of this Section, that f satisfies (A_1) and (A_2).

Lemma 4 *Let ω be a smooth bounded domain in $\mathbf{R}^N$. Assume p, q, r are $C^{0,\mu}$-functions on $\overline{\omega}$ such that $r \geq 0$ and $p > 0$ in $\overline{\omega}$. Then for any non-negative function $0 \not\equiv \Phi \in C^{0,\mu}(\partial\omega)$ the boundary value problem*

$$\begin{cases} \Delta u + q(x)u = p(x)f(u) - r(x) & \text{in } \omega, \\ u > 0 & \text{in } \omega, \\ u = \Phi & \text{on } \partial\omega, \end{cases} \tag{18}$$

has a unique solution.

Proof. By Lemma 2, problem (18) has at most a solution. The existence of a positive solution will be obtained by device of sub and super-solutions.

Set $p_0 := \inf_\omega p > 0$. Define $\bar{f}(u) = p_0 f(u) - \|q\|_\infty u - \bar{r}$, where $\bar{r} := \sup_\omega r + 1 > 0$. Let t_1 be the unique positive solution of the equation $\bar{f}(u) = 0$. By Remark 3 we derive that $\lim_{u\to\infty} \frac{\bar{f}(u)}{f(u)} = p_0 > 0$. Combining this with (A_2), we conclude that the function $\varphi(w) = \bar{f}(w + t_1)$ defined for $w \geq 0$ satisfies the assumptions of Theorem III in [20]. It follows that there exists a positive large solution for the equation $\Delta w = \varphi(w)$ in ω. Thus the function $\bar{u}(x) = w(x) + t_1$, for all $x \in \omega$, is a positive large solution of the problem

$$\Delta u + \|q\|_\infty u = p_0 f(u) - \bar{r} \quad \text{in } \omega. \tag{19}$$

Applying Theorem 4 in the Appendix, the boundary value problem

$$\begin{cases} \Delta u = \|q\|_\infty u + \|p\|_\infty f(u) & \text{in } \omega, \\ u > 0 & \text{in } \omega, \\ u = \Phi & \text{on } \partial\omega, \end{cases} \tag{20}$$

has a unique classical solution $\underline{u}$. By Lemma 2, we find that $\underline{u} \leq \overline{u}$ in ω and $\underline{u}$ (resp., $\overline{u}$) is a positive sub-solution (resp., super-solution) of problem (18). It follows that (18) has a unique solution. ■

Under the assumptions of Lemma 4 we obtain the following result which generalizes [26, Lemma 1.3].

Corollary 1 *There exists a positive large solution of the problem*

$$\Delta u + q(x)u = p(x)f(u) - r(x) \qquad \text{in } \omega. \tag{21}$$

Proof. Set $\Phi = n$ and let u_n be the unique solution of (18). By Lemma 2, $u_n \leq u_{n+1} \leq \overline{u}$ in ω, where $\overline{u}$ denotes a large solution of (19). Thus $\lim_{n\to\infty} u_n(x) = u_\infty(x)$ exists and is a positive large solution of (21). Furthermore, every positive large solution of (21) dominates u_∞, i.e., the solution u_∞ is the *minimal large solution*. This follows from the definition of u_∞ and Lemma 2. ■

Lemma 5 *If $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ is a non-negative function and $b > 0$ on $\partial\Omega$, then the boundary value problem*

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \end{cases} \tag{22}$$

has a solution if and only if $a \in (-\infty, \lambda_{\infty,1})$. Moreover, in this case, the solution is unique.

Proof. The first part follows exactly in the same way as the proof of Theorem 1 (necessary condition).

For the sufficient condition, fix $a < \lambda_{\infty,1}$ and let $\lambda_{\infty,1} > \lambda_* > \max\{a, \lambda_1(\mu_0)\}$. Let u_* be the unique positive solution of (E_a) with $a = \lambda_*$.

Let Ω_i $(i = 1, 2)$ be subdomains of Ω such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and $\Omega \setminus \overline{\Omega}_1$ is smooth. We define $u_+ \in C^2(\Omega)$ as a positive function in Ω such that $u_+ \equiv u_\infty$ on $\Omega \setminus \Omega_2$ and $u_+ \equiv u_*$ on Ω_1. Here u_∞ denotes a positive large solution of (21) for $p(x) = b(x)$, $r(x) = 0$, $q(x) = a$ and $\omega = \Omega \setminus \overline{\Omega}_1$. Using Remark 3 and the fact that $b_0 := \inf_{\Omega_2\setminus\Omega_1} b$ is positive, it is easy to check that if $C > 0$ is large enough then $\overline{v}_\Phi = Cu_+$ satisfies

$$\begin{cases} \Delta\overline{v}_\Phi + a\overline{v}_\Phi \leq b(x)f(\overline{v}_\Phi) & \text{in } \Omega\,, \\ \overline{v}_\Phi = \infty & \text{on } \partial\Omega\,. \\ \overline{v}_\Phi \geq \max\limits_{\partial\Omega} \Phi & \text{in } \Omega\,. \end{cases}$$

By Theorem 4 in the Appendix, there exists a unique classical solution $\underline{v}_\Phi$ of the problem

$$\begin{cases} \Delta\underline{v}_\Phi = |a|\underline{v}_\Phi + \|b\|_\infty f(\underline{v}_\Phi) & \text{in } \Omega, \\ \underline{v}_\Phi > 0 & \text{in } \Omega, \\ \underline{v}_\Phi = \Phi & \text{on } \partial\Omega\,. \end{cases}$$

It is clear that $\underline{v}_\Phi$ is a positive sub-solution of (22) and $\underline{v}_\Phi \leq \max_{\partial\Omega}\Phi \leq \overline{v}_\Phi$ in Ω. Therefore, by the sub-super solution method, problem (22) has at least a solution v_Φ between $\underline{v}_\Phi$ and $\overline{v}_\Phi$. Next, the uniqueness of solution to (22) can be obtained by using essentially the same technique as in [10, Theorem 1] or [9, Appendix II]. ∎

Proof of Theorem 1 completed. - Fix $a \in (-\infty, \lambda_{\infty,1})$. Two cases may occur:

CASE 1: $b > 0$ on $\partial\Omega$. Denote by v_n the unique solution of (22) with $\Phi \equiv n$. For $\Phi \equiv 1$, set $v := \underline{v}_\Phi$ and $V := \overline{v}_\Phi$, where $\underline{v}_\Phi$ and $\overline{v}_\Phi$ are defined in the proof of Lemma 5. The sub and super-solutions method combined with the uniqueness of solution of (22) shows that $v \leq v_n \leq v_{n+1} \leq V$ in Ω. Hence $v_\infty(x) := \lim_{n\to\infty} v_n(x)$ exists and is a positive large solution of (1).

CASE 2: $b \geq 0$ on $\partial\Omega$. Let z_n $(n \geq 1)$ be the unique solution of (18) for $p \equiv b + 1/n$, $r \equiv 0$, $q \equiv a$, $\Phi \equiv n$ and $\omega = \Omega$. By Lemma 1, (z_n) is non-decreasing. Moreover, (z_n) is uniformly bounded on every compact subdomain of Ω. Indeed, if $K \subset \Omega$ is an arbitrary compact set, then $d := \text{dist}\,(K, \partial\Omega) > 0$. Choose $\delta \in (0, d)$ small enough so that $\overline{\Omega}_0 \subset C_\delta$, where $C_\delta = \{x \in \Omega :\ \text{dist}\,(x, \partial\Omega) > \delta\}$. Since $b > 0$ on ∂C_δ, Case 1 allows us to define z_+ as a positive large solution of (1) for $\Omega = C_\delta$. Using Lemma 1 for $p \equiv b + 1/n$ and $\omega = C_\delta$ we obtain $z_n \leq z_+$ in C_δ, for all $n \geq 1$. So, (z_n) is uniformly bounded on K. By the monotonicity of (z_n), we conclude that $z_n \to \underline{z}$ in $L^\infty_{\text{loc}}(\Omega)$. Finally, standard elliptic regularity arguments lead to $z_n \to \underline{z}$ in $C^{2,\mu}(\Omega)$. This completes the proof of Theorem 1. ∎

4 Auxiliary results

The main purpose of this Section is to provide an equivalent criterion to the Keller-Osserman condition (A_2). To our best knowledge there are no results of this type. We point out that, throughout this Section, a significant role plays the set $\mathcal{G}$ defined by

$$\mathcal{G} = \left\{ g :\ \ \exists \delta > 0 \text{ such that } g \in C^2(0,\delta),\ \ g'' > 0 \text{ on } (0,\delta),\ \ \lim_{t\searrow 0} g(t) = \infty \ \text{ and } \ \exists \lim_{t\searrow 0} \frac{g'(t)}{g''(t)} \right\}.$$

Note that $\mathcal{G} \not\equiv \emptyset$. We see, for example, that $e^\Theta \subset \mathcal{G}$ where

$$\Theta = \left\{ \theta :\ \ \theta \in C^2(0,\infty),\ \ \theta \text{ is convex on } (0,\infty) \ \text{ and } \ \lim_{t\searrow 0} \theta(t) = \infty \right\}.$$

Obviously, $\Theta \not\equiv \emptyset$. Let $\theta \in \Theta$ be arbitrary. Since θ' is non-decreasing on $(0,\infty)$ and $\lim_{t\searrow 0} \theta(t) = \infty$, it follows that $\lim_{t\searrow 0} \theta'(t) = -\infty$. Then,

$$\left| \frac{\theta'(t)}{(\theta'(t))^2 + \theta''(t)} \right| \leq \frac{1}{|\theta'(t)|} \to 0 \quad \text{as } t \searrow 0$$

which proves that $e^\theta \in \mathcal{G}$.

Remark 4 $\displaystyle\lim_{t\searrow 0} \frac{g(t)}{g''(t)} = \lim_{t\searrow 0} \frac{g'(t)}{g''(t)} = 0$ *for any function* $g \in \mathcal{G}$.

Indeed, if $g \in \mathcal{G}$ is chosen arbitrarily, then

$$\lim_{t\searrow 0} g'(t) = -\infty, \quad \lim_{t\searrow 0} \ln g(t) = \infty \quad \text{and} \quad \lim_{t\searrow 0} \ln |g'(t)| = \infty. \tag{23}$$

L'Hospital's rule and (23) imply that $\lim_{t\searrow 0} \frac{g(t)}{g'(t)} = \lim_{t\searrow 0} \frac{g'(t)}{g''(t)} = 0$.

Lemma 6 *Assume* $(\tilde{A}_1)$. *Then, the following hold:*
(i) $\gamma \geq 0$.
(ii) $\gamma \leq 1/2$ *provided that* (A_2) *is fulfilled.*

Proof. (i) If we suppose $\gamma < 0$, then there exists $s_1 > 0$ such that

$$\left(\frac{F}{f}\right)'(u) \leq \frac{\gamma}{2} < 0 \qquad \text{for any } u \geq s_1.$$

Integrating this inequality over (s_1, ∞) we obtain a contradiction. It follows that $\gamma \geq 0$.

(ii) Let (A_2) be satisfied. Using the definition of γ, we find $\lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)} = 1 - \gamma$. By Remark 3 and L'Hospital's rule we obtain

$$\lim_{u\to\infty} \frac{F(u)}{f^2(u)} \overset{\frac{\infty}{\infty}}{=} \lim_{u\to\infty} \frac{1}{2f'(u)} = 0$$

and

$$0 \leq \lim_{u\to\infty} \frac{\frac{\sqrt{F(u)}}{f(u)}}{\int_u^\infty \frac{ds}{\sqrt{F(s)}}} \overset{\frac{0}{0}}{=} -\frac{1}{2} + \lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)} = \frac{1}{2} - \gamma. \tag{24}$$

This concludes our proof. ■

Lemma 7 *Assume* $(\tilde{A}_1)$. *Then the Keller-Osserman growth condition* (A_2) *holds if and only if*

$$(A_g) \quad \lim_{t\searrow 0} \frac{tf(g(t))}{g''(t)} = \infty \quad \textit{for some function } g \in \mathcal{G}\,.$$

Proof. A. Necessary condition. Since (A_2) holds, we can define the positive function g as follows

$$\int_{g(t)}^\infty \frac{ds}{\sqrt{F(s)}} = t^\vartheta \quad \text{for all } t > 0, \quad \text{where } \vartheta \in \left(\frac{3}{2}, \infty\right) \text{ is arbitrary.} \tag{25}$$

Obviously, $g \in C^2(0,\infty)$ and $\lim_{t\searrow 0} g(t) = \infty$. We claim that $g \in \mathcal{G}$ and condition (A_g) is fulfilled. To argue this, we divide our argument into three steps:

Step 1: $\displaystyle\lim_{t\searrow 0} \frac{g'(t)}{t^{2\vartheta-1}f(g(t))} = \vartheta\left(\gamma - \frac{1}{2}\right)$.
We derive twice relation (25) and obtain

$$g'(t) = -\vartheta t^{\vartheta-1}\sqrt{F(g(t))} \tag{26}$$

$$g''(t) = \frac{\vartheta-1}{t}g'(t) + \frac{\vartheta^2}{2}t^{2\vartheta-2}f(g(t)) = \frac{\vartheta^2}{2}t^{2\vartheta-2}f(g(t))\left(\frac{2(\vartheta-1)}{\vartheta^2}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))} + 1\right). \tag{27}$$

By using (26) and (24) we find

$$\lim_{t\searrow 0} \frac{g'(t)}{t^{2\vartheta-1}f(g(t))} = \lim_{t\searrow 0} \frac{-\vartheta t^{\vartheta-1}\sqrt{F(g(t))}}{t^{2\vartheta-1}f(g(t))} = \lim_{t\searrow 0} -\vartheta\frac{\frac{\sqrt{F(g(t))}}{f(g(t))}}{\int_{g(t)}^\infty \frac{ds}{\sqrt{F(s)}}} = \lim_{u\to\infty} -\vartheta\frac{\frac{\sqrt{F(u)}}{f(u)}}{\int_u^\infty \frac{ds}{\sqrt{F(s)}}} = \vartheta\left(\gamma - \frac{1}{2}\right).$$

Step 2: $g'' > 0$ on $(0,\delta)$ for δ small enough.

Since $\gamma \geq 0$, by using Step 1 we find

$$\lim_{t \searrow 0} \frac{2(\vartheta - 1)}{\vartheta^2} \frac{g'(t)}{t^{2\vartheta-1} f(g(t))} = \frac{2(\vartheta - 1)}{\vartheta} \left(\gamma - \frac{1}{2}\right) \geq \frac{1}{\vartheta} - 1 > -1 \tag{28}$$

In view of (27), the assertion of this step follows.

STEP 3: $\lim_{t \searrow 0} \frac{g'(t)}{g''(t)} = 0$ and $\lim_{t \searrow 0} \frac{tf(g(t))}{g''(t)} = \infty.$

Taking into account (27) and (28) we find

$$\lim_{t \searrow 0} \frac{g'(t)}{g''(t)} = \lim_{t \searrow 0} \frac{2t}{\vartheta^2} \frac{g'(t)}{t^{2\vartheta-1} f(g(t))} \frac{1}{\frac{2(\vartheta-1)}{\vartheta^2} \frac{g'(t)}{t^{2\vartheta-1} f(g(t))} + 1} = 0$$

and, for any $t \in (0, \delta)$ where $\delta > 0$ is given by Step 2, we have

$$\frac{tf(g(t))}{g''(t)} = \frac{tf(g(t))}{\frac{\vartheta-1}{t} g'(t) + \frac{\vartheta^2}{2} t^{2\vartheta-2} f(g(t))} \geq \frac{tf(g(t))}{\frac{\vartheta^2}{2} t^{2\vartheta-2} f(g(t))} = \frac{2}{\vartheta^2 t^{2\vartheta-3}}.$$

Sending t to 0, the claim of Step 3 is proved.

B. SUFFICIENT CONDITION. Let $g \in \mathcal{G}$ be chosen so that (A_g) is fulfilled. By L'Hospital's rule we find

$$\lim_{t \searrow 0} \frac{(g'(t))^2}{F(g(t))} = 2 \lim_{t \searrow 0} \frac{g''(t)}{f(g(t))} = 0.$$

We choose $\delta > 0$ small enough such that $g'(s) < 0$ and $g''(s) > 0$ for all $s \in (0, \delta)$. It follows that

$$\int_{g(\delta)}^{\infty} \frac{dt}{\sqrt{F(t)}} = \lim_{t \searrow 0} \int_{g(\delta)}^{g(t)} \frac{ds}{\sqrt{F(s)}} = \lim_{t \searrow 0} \int_t^{\delta} \frac{-g'(s)\, ds}{\sqrt{F(g(s))}} \leq \delta \sup_{t \in (0,\delta)} \frac{-g'(t)}{\sqrt{F(g(t))}} < \infty.$$

Hence, the growth condition (A_2) holds. ■

Lemma 8 *Assume that $(\tilde{A}_1)$ with $\gamma \neq 0$, (A_2), (B_1) and (B_2) are fulfilled. Then, the following hold:*
(i) $K'(0)(1 - 2\gamma) + 2\gamma \in (0, 1]$.
(ii) $h \in \mathcal{G}$, *where h is the function defined by (7).*

Proof. (i) Since $\gamma \neq 0$, by Lemma 6 we find $0 < \gamma \leq 1/2$. Therefore, the claim of (i) follows if we prove that $K'(0) \in [0, 1]$. To this aim, we remark that $K(0) = 0$. Suppose that $K(0) \neq 0$. Then, we obtain

$$\lim_{t \searrow 0} \left[\ln\left(\int_0^t \sqrt{k(s)}\, ds\right)\right]'(t) = \frac{1}{K(0)} \in (0, \infty),$$

which contradicts the fact that $\lim_{t \searrow 0} \ln\left(\int_0^t \sqrt{k(s)}\, ds\right) = -\infty$. So, $K(0) = 0$. This produces $K'(0) \geq 0$. Since $K \in C_1[0, \delta_0)$, we have

$$K'(0) = \lim_{t \searrow 0} \left(\frac{\int_0^t \sqrt{k(s)}\, ds}{\sqrt{k(t)}}\right)'$$

so that

$$\lim_{t\searrow 0}\frac{k'(t)\int_0^t\sqrt{k(s)}\,ds}{k^{3/2}(t)}=2\left(1-\lim_{t\searrow 0}\left(\frac{\int_0^t\sqrt{k(s)}\,ds}{\sqrt{k(t)}}\right)'\right)=2(1-K'(0)). \tag{29}$$

Hence, $K'(0)\le 1$. Indeed, assuming the contrary, relation (29) yields $k'(t)<0$ for $t\in(0,\tilde{\delta})$ for some $0<\tilde{\delta}<\delta_0$. But this is impossible, since $\lim_{t\searrow 0}k(t)=0$ and $k>0$ on $(0,\delta_0)$.

(ii) Using the definition of h, we deduce that $h\in C^2(0,\delta_0)$ and $\lim_{t\searrow 0}h(t)=\infty$. Then, by twice deriving relation (7), we find

$$h'(t)=-\sqrt{k(t)}\,\sqrt{2F(h(t))},\qquad \forall t\in(0,\delta_0),$$

respectively,

$$\begin{aligned} h''(t)&=k(t)f(h(t))-\frac{1}{\sqrt{2}}\frac{\sqrt{F(h(t))}}{\sqrt{k(t)}}\,k'(t)\\ &=k(t)f(h(t))\left(1-\frac{k'(t)\int_0^t\sqrt{k(s)}\,ds}{k^{3/2}(t)}\,\frac{\frac{\sqrt{F(h(t))}}{f(h(t))}}{\int_{h(t)}^{\infty}\frac{ds}{\sqrt{F(s)}}}\right). \end{aligned}$$

Using (24) and (29), we obtain

$$\lim_{t\searrow 0}\frac{h'(t)}{h''(t)}=\frac{-2}{K'(0)(1-2\gamma)+2\gamma}\lim_{t\searrow 0}\frac{\frac{\sqrt{F(h(t))}}{f(h(t))}}{\int_{h(t)}^{\infty}\frac{ds}{\sqrt{F(s)}}}\lim_{t\searrow 0}\frac{\int_0^t\sqrt{k(s)}\,ds}{\sqrt{k(t)}}=\frac{2\gamma-1}{K'(0)(1-2\gamma)+2\gamma}K(0)=0.$$

and

$$\lim_{t\searrow 0}\frac{h''(t)}{k(t)f(h(t))}=K'(0)(1-2\gamma)+2\gamma>0 \tag{30}$$

which shows that h'' is positive on $(0,\delta_1)$ for some $\delta_1>0$. This concludes our proof. ■

5 Proof of Theorem 2

We start with the following result.

Lemma 9 *Assume $b>0$ on $\partial\Omega$. If (A_1) and (A_2) hold, then for any positive function $\Phi\in C^{2,\mu}(\partial\Omega_0)$ and $a\in\mathbf{R}$ the problem*

$$\begin{cases} \Delta u+au=b(x)f(u) & \text{in } \Omega\setminus\overline{\Omega}_0\,,\\ \mathcal{B}u=0 & \text{on } \partial\Omega\,,\\ u=\Phi & \text{on } \partial\Omega_0\,, \end{cases} \tag{31}$$

has a unique positive solution.

Proof. In view of Lemma 3 we find that (31) has at most a positive solution. To prove the existence of a positive solution to (31) we shall use the sub and super-solution method.

Let $\omega \subset\subset \Omega_0$ be such that the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\Omega_0 \setminus \overline{\omega}$ is greater than a. Let $p \in C^{0,\mu}(\overline{\Omega})$ be such that $p(x) = b(x)$ for $x \in \overline{\Omega} \setminus \Omega_0$, $p(x) = 0$ for $x \in \overline{\Omega}_0 \setminus \omega$ and $p(x) > 0$ for $x \in \omega$. By virtue of Lemma 5, problem

$$\begin{cases} \Delta u + au = p(x)f(u) & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega, \end{cases}$$

has a unique positive solution u_1.

We choose Ω_1 and Ω_2 two subdomains of Ω such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$.
Define $u^* \in C^2(\overline{\Omega} \setminus \Omega_0)$ so that $u^* \equiv 1$ on $\overline{\Omega} \setminus \Omega_2$, $u^* \equiv u_1$ on $\overline{\Omega}_1 \setminus \Omega_0$ and $m_* := \min_{\overline{\Omega} \setminus \Omega_0} u^* > 0$.

CLAIM: *For $\ell \geq 1$ large enough, ℓu^* is a super-solution for problem (31).*

We first observe that

$$-\Delta(\ell u^*) = \ell a u_1 - \ell p(x) f(u_1) \geq a(\ell u^*) - b(x) f(\ell u^*) \quad \text{for } x \in \overline{\Omega}_1 \setminus \overline{\Omega}_0 \ \text{ and } \ \ell \geq 1. \tag{32}$$

Denote by $M^* := \sup_{\Omega \setminus \Omega_1} (au^* + \Delta u^*)$ and $b_0 := \min_{\overline{\Omega} \setminus \Omega_1} b > 0$. By Remark 3, we obtain that there exists $\ell_1 \geq 1$ such that

$$f(\ell m_*) \geq \frac{\ell M^*}{b_0} \quad \text{for all } \ell \geq \ell_1.$$

For $x \in \Omega \setminus \overline{\Omega}_1$ and $\ell \geq \ell_1$ we have

$$b(x) f(\ell u^*) \geq b_0 f(\ell m_*) \geq \ell(au^* + \Delta u^*)$$

which can be rewritten as

$$-\Delta(\ell u^*) \geq a(\ell u^*) - b(x) f(\ell u^*) \quad \text{for } x \in \Omega \setminus \overline{\Omega}_1 \ \text{ and } \ell \geq \ell_1. \tag{33}$$

By (32) and (33) it follows that

$$-\Delta(\ell u^*) \geq a(\ell u^*) - b(x) f(\ell u^*) \quad \text{in } \Omega \setminus \overline{\Omega}_0, \ \text{ for any } \ \ell \geq \ell_1.$$

On the other hand,

$$\mathcal{B}(\ell u^*) \geq \ell \min\{1, \min_{x \in \partial\Omega} \beta(x)\} \geq 0 \quad \text{on } \partial\Omega, \ \text{ for every } \ell > 0.$$

By taking $\ell \geq \max\{\max_{\partial\Omega_0} \Phi / m_*; \ell_1\}$ the claim follows.

Set $\overline{b} := \sup_\Omega b$. By Theorem 4 in the Appendix, the boundary value problem

$$\begin{cases} \Delta u_* = \overline{b} f(u_*) + |a| u_* & \text{in } \Omega \setminus \overline{\Omega}_0\,, \\ u_* = 0 & \text{on } \partial\Omega\,, \\ u_* = \Phi & \text{on } \partial\Omega_0\,, \end{cases} \tag{34}$$

has a unique non-negative solution, which is positive in $\Omega \setminus \overline{\Omega}_0$. Since $u_* = 0$ on $\partial\Omega$ we find that $\mathcal{R} u_* = \partial_\nu u_* \leq 0$ on $\partial\Omega$. It is easy to see that u_* is a sub-solution of (31) and $u_* \leq \ell u^*$ in $\overline{\Omega} \setminus \Omega_0$ for ℓ large enough. The conclusion of Lemma 9 follows now by the sub-super solution method. ■

Corollary 2 *If Ω_0 is replaced by Ω_m defined in (5), then the statement of Lemma 9 holds.*

Proof. The proof is very easy in this case. The construction of the sub-solution is made as before, while the super-solution can be chosen any number $\ell \geq 1$ large enough. ■

We now come back to the proof of Theorem 2, that will be divided into two steps:

Step 1. Existence of the minimal positive solution for problem (4).

For any $n \geq 1$, let u_n be the unique positive solution of problem (31) with $\Phi \equiv n$. By Lemma 3, $u_n(x)$ increases with n for all $x \in \overline{\Omega} \setminus \overline{\Omega}_0$. Moreover, we prove

Lemma 10 *The sequence $(u_n(x))_n$ is bounded from above by some function $V(x)$ which is uniformly bounded on all compact subsets of $\overline{\Omega} \setminus \overline{\Omega}_0$.*

Proof. Let b^* be a C^2-function on $\overline{\Omega} \setminus \Omega_0$ such that

$$0 < b^*(x) \leq b(x) \quad \forall x \in \overline{\Omega} \setminus \overline{\Omega}_0.$$

For x bounded away from $\partial\Omega_0$ is not a problem to find such a function b^*. For x satisfying $0 < d(x) < \delta$ with $\delta > 0$ small such that $x \to d(x)$ is a C^2-function, we can take

$$b^*(x) = \int_0^{d(x)} \int_0^t [\min_{d(z) \geq s} b(z)]\, ds\, dt.$$

Let $g \in \mathcal{G}$ be a function such that (A_g) holds. The existence of g is guaranteed by Lemma 7. Since $b^*(x) \to 0$ as $d(x) \searrow 0$, we deduce, by Remark 4 and (A_1), the existence of some $\delta > 0$ such that for all $x \in \Omega$ with $0 < d(x) < \delta$ and $\xi > 1$

$$\frac{b^*(x) f(g(b^*(x))\xi)}{g''(b^*(x))\, \xi} > \sup_{\overline{\Omega} \setminus \Omega_0} |\nabla b^*|^2 + \frac{g'(b^*(x))}{g''(b^*(x))} \inf_{\overline{\Omega} \setminus \Omega_0} (\Delta b^*) + a\, \frac{g(b^*(x))}{g''(b^*(x))}.$$

Here, $\delta > 0$ is taken sufficiently small so that $g'(b^*(x)) < 0$ and $g''(b^*(x)) > 0$ for all x with $0 < d(x) < \delta$.

For $n_0 \geq 1$ fixed, define V^* as follows
(i) $V^*(x) = u_{n_0}(x) + 1$ for $x \in \overline{\Omega}$ and near $\partial\Omega$;
(ii) $V^*(x) = g(b^*(x))$ for x satisfying $0 < d(x) < \delta$;
(iii) $V^* \in C^2(\overline{\Omega} \setminus \overline{\Omega}_0)$ is positive on $\overline{\Omega} \setminus \overline{\Omega}_0$.

We show that for $\xi > 1$ large enough the upper bound of the sequence $(u_n(x))_n$ can be taken as $V(x) = \xi V^*(x)$. Since

$$\mathcal{B}V(x) = \xi\, \mathcal{B}V^*(x) \geq \xi \min\{1, \beta(x)\} \geq 0, \quad \forall x \in \partial\Omega \quad \text{and} \quad \lim_{d(x) \searrow 0} [u_n(x) - V(x)] = -\infty < 0,$$

to conclude that $u_n(x) \leq V(x)$ for all $x \in \overline{\Omega} \setminus \overline{\Omega}_0$ it is sufficient to show, by virtue of Lemma 3, that

$$-\Delta V(x) \geq aV(x) - b(x) f(V(x)), \qquad \forall x \in \Omega \setminus \overline{\Omega}_0. \tag{35}$$

For $x \in \Omega$ satisfying $0 < d(x) < \delta$ and $\xi > 1$ we have

$$\begin{aligned} -\Delta V(x) - aV(x) + b(x) f(V(x)) &= -\xi \Delta g(b^*(x)) - a\, \xi g(b^*(x)) + b(x) f(g(b^*(x))\xi) \\ &\geq \xi g''(b^*(x)) \left(-\frac{g'(b^*(x))}{g''(b^*(x))}\, \Delta b^*(x) - |\nabla b^*(x)|^2 - a\, \frac{g(b^*(x))}{g''(b^*(x))} + b^*(x)\, \frac{f(g(b^*(x))\xi)}{g''(b^*(x))\, \xi} \right) > 0. \end{aligned}$$

For $x \in \Omega$ satisfying $d(x) \geq \delta$,

$$-\Delta V(x) - aV(x) + b(x)f(V(x)) = \xi \left(-\Delta V^*(x) - aV^*(x) + b(x)\,\frac{f(\xi V^*(x))}{\xi} \right) \geq 0$$

for ξ sufficiently large. In the last inequality, we have used (iii) and Remark 3. It follows that (35) is fulfilled provided ξ is large enough. This finishes the proof of the lemma. ■

By Lemma 10, $\underline{U}_a(x) \equiv \lim_{n\to\infty} u_n(x)$ exists, for any $x \in \overline{\Omega} \setminus \overline{\Omega}_0$. Moreover, $\underline{U}_a$ is a positive solution of (4). Using Lemma 3 once more, we find that any positive solution u of (4) satisfies $u \geq u_n$ on $\overline{\Omega} \setminus \overline{\Omega}_0$, for all $n \geq 1$. Hence $\underline{U}_a$ is the minimal positive solution of (4).

Proof of Theorem 2 completed.

Step 2. Existence of the maximal positive solution for problem (4).

Lemma 11 *If Ω_0 is replaced by Ω_m defined in (5), then problem (4) has a minimal positive solution provided that (A_1) and (A_2) are fulfilled.*

Proof. The argument used here (more easier, since $b > 0$ on $\overline{\Omega} \setminus \Omega_m$) is similar to that in Step 1. The only difference which appears in the proof (except the replacement of Ω_0 by Ω_m) is related to the construction of $V^*(x)$ for x near $\partial\Omega_m$. Here, instead of Lemma 7 we use our Theorem 1 which says that, for any $a \in \mathbf{R}$, there exists a positive large solution $u_{a,\infty}$ of problem (1) in the domain $\Omega \setminus \overline{\Omega}_m$. We define $V^*(x) = u_{a,\infty}(x)$ for $x \in \Omega \setminus \overline{\Omega}_m$ and near $\partial\Omega_m$. For $\xi > 1$ and $x \in \Omega \setminus \overline{\Omega}_m$ near $\partial\Omega_m$ we have

$$\begin{aligned}
-\Delta V(x) - aV(x) + b(x)f(V(x)) &= -\xi\Delta V^*(x) - a\xi V^*(x) + b(x)f(\xi V^*(x)) \\
&= b(x)[f(\xi V^*(x)) - \xi f(V^*(x)] \geq 0.
\end{aligned}$$

This completes the proof. ■

Let v_m be the minimal positive solution for the problem considered in the statement of Lemma 11. By Lemma 3, $v_m \geq v_{m+1} \geq u$ on $\overline{\Omega} \setminus \overline{\Omega}_m$, where u is any positive solution of (4). Hence $\overline{U}_a(x) := \lim_{m\to\infty} v_m(x) \geq u(x)$. A regularity and compactness argument shows that $\overline{U}_a$ is a positive solution of (4). Consequently, $\overline{U}_a$ is the maximal positive solution. This concludes the proof of Theorem 2. ■

6 Proof of Theorem 3

By (A_4) we deduce that the mapping $(0,\infty) \ni \xi \longmapsto A(\xi) = \lim_{u\to\infty} \dfrac{f(\xi u)}{\xi f(u)}$ is a continuous positive function, since $A(1/\xi) = 1/A(\xi)$ for any $\xi \in (0,1)$. Moreover, we claim

Lemma 12 *The function $A : (0,\infty) \to (0,\infty)$ is bijective, provided that (A_3) and (A_4) are fulfilled.*

Proof. By the continuity of A, we see that the surjectivity of A follows if we prove that $\lim_{\xi\searrow 0} A(\xi) = 0$. To this aim, let $\xi \in (0,1)$ be fixed. Using (A_3) we find

$$\frac{f(\xi u)}{\xi f(u)} \leq \xi^{\zeta}, \quad \forall u \geq \frac{t_0}{\xi}$$

which yields $A(\xi) \leq \xi^{\zeta}$. Since $\xi \in (0,1)$ is arbitrary, it follows that $\lim_{\xi \searrow 0} A(\xi) = 0$.

We now prove that the function $\xi \longmapsto A(\xi)$ is increasing on $(0,\infty)$ which concludes our lemma. Let $0 < \xi_1 < \xi_2 < \infty$ be chosen arbitrarily. Using assumption (A_3) once more, we obtain

$$f(\xi_1 u) = f\left(\frac{\xi_1}{\xi_2}\xi_2 u\right) \leq \left(\frac{\xi_1}{\xi_2}\right)^{1+\zeta} f(\xi_2 u), \quad \forall u \geq t_0 \,\frac{\xi_2}{\xi_1}.$$

It follows that

$$\frac{f(\xi_1 u)}{\xi_1 f(u)} \leq \left(\frac{\xi_1}{\xi_2}\right)^{\zeta} \frac{f(\xi_2 u)}{\xi_2 f(u)}, \quad \forall u \geq t_0 \,\frac{\xi_2}{\xi_1}.$$

Passing to the limit as $u \to \infty$ we find

$$A(\xi_1) \leq \left(\frac{\xi_1}{\xi_2}\right)^{\zeta} A(\xi_2) < A(\xi_2),$$

which finishes the proof. ■

Proof of Theorem 3 completed.

By Lemma 8, $h \in \mathcal{G}$. Set $\Pi(\xi) = \lim_{d(x)\searrow 0} b(x)\,\dfrac{f(h(d(x))\xi)}{h''(d(x))\,\xi}$, for any $\xi > 0$. Using (B_1) and (30), we find

$$\begin{aligned}
\Pi(\xi) &= \lim_{d(x)\searrow 0} \frac{b(x)}{k(d(x))}\,\frac{k(d(x))f(h(d(x)))}{h''(d(x))}\,\frac{f(h(d(x))\xi)}{\xi f(h(d(x)))} = c\,\lim_{t\searrow 0}\frac{k(t)f(h(t))}{h''(t)}\,\lim_{u\to\infty}\frac{f(\xi u)}{\xi f(u)} \\
&= \frac{c}{K'(0)(1-2\gamma)+2\gamma}\,A(\xi).
\end{aligned}$$

This and Lemma 12 imply that the function $\Pi : (0,\infty) \to (0,\infty)$ is bijective. Let ξ_0 be the unique positive solution of $\Pi(\xi) = 1$, that is $A(\xi_0) = \dfrac{K'(0)(1-2\gamma)+2\gamma}{c}$.

For $\varepsilon \in (0, 1/4)$ arbitrary, we denote $\xi_1 = \Pi^{-1}(1-4\varepsilon)$, respectively $\xi_2 = \Pi^{-1}(1+4\varepsilon)$.

Using Remark 4, (B_1) and the regularity of $\partial\Omega_0$, we can choose $\delta > 0$ small enough such that

(i) $\operatorname{dist}(x, \partial\Omega_0)$ is a C^2 function on the set $\{x \in \Omega :\ \operatorname{dist}(x,\partial\Omega_0) \leq 2\delta\}$;

(ii) $\left|\dfrac{h'(s)}{h''(s)}\,\Delta d(x) + a\,\dfrac{h(s)}{h''(s)}\right| < \varepsilon$ and $h''(s) > 0$ for all $s \in (0, 2\delta)$ and x satisfying $0 < d(x) < 2\delta$;

(iii) $(\Pi(\xi_2) - \varepsilon)\,\dfrac{h''(d(x))\,\xi_2}{f(h(d(x))\xi_2)} \leq b(x) \leq (\Pi(\xi_1)+\varepsilon)\,\dfrac{h''(d(x))\,\xi_1}{f(h(d(x))\xi_1)}$, for every x with $0 < d(x) < 2\delta$.

(iv) $b(y) < (1+\varepsilon)b(x)$, for every x, y with $0 < d(y) < d(x) < 2\delta$.

Let $\sigma \in (0,\delta)$ be arbitrary. We define $\underline{v}_\sigma(x) = h(d(x)+\sigma)\xi_1$, for any x with $d(x)+\sigma < 2\delta$, respectively $\overline{v}_\sigma(x) = h(d(x)-\sigma)\xi_2$ for any x with $\sigma < d(x) < 2\delta$.

Using (ii), (iv) and the first inequality in (iii), when $\sigma < d(x) < 2\delta$, we obtain (since $|\nabla d(x)| \equiv 1$)

$$\begin{aligned}
&-\Delta\overline{v}_\sigma(x) - a\overline{v}_\sigma(x) + b(x)f(\overline{v}_\sigma(x)) \\
&= \xi_2\left(-h'(d(x)-\sigma)\,\Delta d(x) - h''(d(x)-\sigma) - a\,h(d(x)-\sigma) + \frac{b(x)f(h(d(x)-\sigma)\xi_2)}{\xi_2}\right) \\
&= \xi_2\,h''(d(x)-\sigma)\left(-\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\,\Delta d(x) - a\,\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} - 1 + \frac{b(x)f(h(d(x)-\sigma)\xi_2)}{h''(d(x)-\sigma)\,\xi_2}\right) \\
&\geq \xi_2\,h''(d(x)-\sigma)\left(-\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\,\Delta d(x) - a\,\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} - 1 + \frac{\Pi(\xi_2)-\varepsilon}{1+\varepsilon}\right) \geq 0
\end{aligned}$$

for all x satisfying $\sigma < d(x) < 2\delta$.

Similarly, using (ii), (iv) and the second inequality in (iii), when $d(x)+\sigma < 2\delta$ we find

$$\begin{aligned}
&-\Delta \underline{v}_\sigma(x) - a\underline{v}_\sigma(x) + b(x) f(\underline{v}_\sigma(x)) \\
&\quad = \xi_1\, h''(d(x)+\sigma)\left(-\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\,\Delta d(x) - a\,\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} - 1 + \frac{b(x) f(h(d(x)+\sigma)\xi_1)}{h''(d(x)+\sigma)\,\xi_1}\right) \\
&\quad \le \xi_1\, h''(d(x)+\sigma)\left(-\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\,\Delta d(x) - a\,\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} - 1 + (1+\varepsilon)(\Pi(\xi_1)+\varepsilon)\right) \le 0,
\end{aligned}$$

for all x satisfying $d(x)+\sigma < 2\delta$.

Define $\Omega_\delta \equiv \{x \in \Omega :\ d(x) < \delta\}$. Let $\omega \subset\subset \Omega_0$ be such that the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\Omega_0 \setminus \overline{\omega}$ is strictly greater than a. Denote by w a positive large solution to the following problem

$$-\Delta w = aw - p(x) f(w) \qquad \text{in } \Omega_\delta\,,$$

where $p \in C^{0,\mu}(\overline{\Omega}_\delta)$ satisfies $0 < p(x) \le b(x)$ for $x \in \overline{\Omega}_\delta \setminus \overline{\Omega}_0$, $p(x) = 0$ on $\overline{\Omega}_0 \setminus \omega$ and $p(x) > 0$ for $x \in \omega$. The existence of w is guaranteed by our Theorem 1.

Suppose that u is an arbitrary solution of (4) and let $v := u + w$. Then v satisfies

$$-\Delta v \ge av - b(x) f(v) \quad \text{in } \Omega_\delta \setminus \overline{\Omega}_0.$$

Since

$$v|_{\partial\Omega_0} = \infty > \underline{v}_\sigma|_{\partial\Omega_0} \quad \text{and} \quad v|_{\partial\Omega_\delta} = \infty > \underline{v}_\sigma|_{\partial\Omega_\delta},$$

by Lemma 3 we find

$$u + w \ge \underline{v}_\sigma \ \text{ on } \Omega_\delta \setminus \overline{\Omega}_0. \tag{36}$$

Similarly

$$\overline{v}_\sigma + w \ge u \ \text{ on } \Omega_\delta \setminus \overline{\Omega}_\sigma. \tag{37}$$

Letting $\sigma \to 0$ in (36) and (37), we deduce

$$h(d(x))\,\xi_2 + 2w \ge u + w \ge h(d(x))\,\xi_1, \quad \forall x \in \Omega_\delta \setminus \overline{\Omega}_0.$$

Since w is uniformly bounded on $\partial\Omega_0$, it follows that

$$\xi_1 \le \liminf_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} \le \limsup_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} \le \xi_2. \tag{38}$$

Letting $\varepsilon \to 0$ in (38) and looking at the definition of ξ_1 respectively ξ_2 we find

$$\lim_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} = \xi_0. \tag{39}$$

This behavior of the solution will be speculated in order to prove that problem (4) has a unique solution. Indeed, let u_1, u_2 be two positive solutions of (4). For any $\varepsilon > 0$, denote $\tilde{u}_i = (1+\varepsilon)\,u_i$, $i = 1, 2$. By virtue of (39) we get

$$\lim_{d(x)\searrow 0} \frac{u_1(x) - \tilde{u}_2(x)}{h(d(x))} = \lim_{d(x)\searrow 0} \frac{u_2(x) - \tilde{u}_1(x)}{h(d(x))} = -\varepsilon\,\xi_0 < 0$$

which implies

$$\lim_{d(x)\searrow 0}[u_1(x)-\tilde{u}_2(x)] = \lim_{d(x)\searrow 0}[u_2(x)-\tilde{u}_1(x)] = -\infty.$$

On the other hand, since $\frac{f(u)}{u}$ is increasing for $u>0$, we obtain

$$\begin{aligned} &-\Delta\tilde{u}_i = -(1+\varepsilon)\,\Delta u_i = (1+\varepsilon)\,(a\,u_i - b(x)f(u_i)) \geq a\,\tilde{u}_i - b(x)f(\tilde{u}_i) \quad \text{in } \Omega\setminus\overline{\Omega}_0, \\ &\mathcal{B}\tilde{u}_i = \mathcal{B}u_i = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

So, by Lemma 3,

$$u_1(x)\leq \tilde{u}_2(x), \quad u_2(x)\leq\tilde{u}_1(x), \quad \forall x\in\Omega\setminus\overline{\Omega}_0.$$

Letting $\varepsilon\to 0$, we obtain $u_1\equiv u_2$. The proof of Theorem 3 is complete. ∎

Remark 5 *Assume that f satisfies (A_1) and (A_3). Then problem (1) with $a\equiv 0$, $b\equiv 1$ has a unique large solution $\tilde{u}$. Moreover, $\tilde{u}$ satisfies the asymptotic condition (see [5, Theorems 2.3 and 2.4])*

$$\lim_{\mathrm{dist}\,(x,\partial\Omega)\to 0}\frac{\tilde{u}(x)}{\Gamma(\mathrm{dist}\,(x,\partial\Omega))} = 1,$$

where Γ is the function defined as

$$\int_{\Gamma(t)}^{\infty}\frac{ds}{\sqrt{2F(s)}} = t, \quad \forall t>0.$$

Let $\Omega_1\subset\subset\Omega$ be a connected subdomain, with smooth boundary such that $\overline{\Omega}_0\subset\Omega_1$. Theorem 3 yields

Corollary 3 *Let (A_4) be added to the assumptions of Remark 5. Then, for any $a\in\mathbf{R}$, problem (4) with $b\equiv 1$ on $\partial\Omega_1$ and Ω_0 replaced by Ω_1, has a unique positive solution U_a. Moreover, U_a behaves on $\partial\Omega_1$ exactly in the same manner as $\tilde{u}$ on $\partial\Omega$, i.e.,*

$$\lim_{\mathrm{dist}\,(x,\partial\Omega_1)\to 0}\frac{U_a(x)}{\Gamma(\mathrm{dist}\,(x,\partial\Omega_1))} = 1.$$

Proof. By Remark 1 (a), we can apply the argument of Lemma 11 to deduce the existence of a positive solution for problem considered here. Concerning the uniqueness, we remark that (B_1) and (B_2) are fulfilled by taking $c=1$ and $k\equiv 1$ on $(0,\infty)$. It follows that h defined by (7) coincides with Γ. But $\Gamma'(t) = -\sqrt{2F(\Gamma(t))}$ and $\Gamma''(t) = f(\Gamma(t))$ for any $t\in(0,\infty)$. Thus, we obtain $\Gamma\in\mathcal{G}$ (without calling Lemma 8) and $\Pi(\xi) = A(\xi)$ for all $\xi>0$. So, by Lemma 12, $\Pi:(0,\infty)\to(0,\infty)$ is bijective. From now on, we proceed as in the proof of Theorem 3 remaining only to replace h by Γ and Ω_0 by Ω_1. ∎

7 Appendix

The following result has been applied several times in the paper and it is mentioned without proof in Marcus [25]. For the convenience of the reader we give in what follows a complete proof of this result.

Theorem 4 *Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain. Assume $0 \not\equiv p \in C^{0,\mu}(\overline{\Omega})$ is non-negative and $f \in C^1[0,\infty)$ is a positive, non-decreasing function on $(0,\infty)$ such that $f(0)=0$. If $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ is non-negative, then the boundary value problem*

$$\begin{cases} \Delta u = p(x)f(u) & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{40}$$

has a unique classical solution, which is positive in Ω.

Remark 6 *The conclusion of Theorem 4 has been established in [11, Theorem 5] when Φ is assumed to be positive on $\partial\Omega$. Our approach for proving the positivity of solution was essentially based on this assumption and it fails when the zero set of Φ is non-empty.*

Under the same assumptions on p and f as in the statement of Theorem 4 we have

Corollary 4 (Strong maximum principle.) *Let Ω be a non-empty domain in $\mathbf{R}^N$. If u is a non-negative classical solution of the equation $\Delta u = p(x)f(u)$ in Ω then the following alternative holds: either $u \equiv 0$ in Ω or u is positive in Ω.*

Proof. If $u \not\equiv 0$ in Ω, then there exists $x_0 \in \Omega$ such that $u(x_0) > 0$. We claim that $u > 0$ in Ω. Arguing by contradiction, let us assume that $u(x_1) = 0$ for some $x_1 \in \Omega$. Let $\omega \subset\subset \Omega$ be a bounded smooth domain such that $x_1 \in \omega$ and $x_0 \in \partial\omega$. Set $p_0 := 1 + \sup_\omega p > 0$ and consider the problem

$$\begin{cases} \Delta v = p_0 f(v) & \text{in } \omega, \\ v = u \not\equiv 0 & \text{on } \partial\omega, \\ v \geq 0 & \text{in } \omega. \end{cases} \tag{41}$$

By Theorem 4, this problem has a unique solution v_0 which, moreover, is positive in ω. It is clear that 0 (resp., u) is sub-solution (resp., super-solution) for problem (41). So, there exists a solution v_1 of (41) satisfying $0 \leq v_1 \leq u$. By uniqueness we deduce that $v_1 = v_0 > 0$ in ω. It follows that $u \geq v_0 > 0$ in ω. But this is impossible since $u(x_1) = 0$. ■

Corollary 5 *Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain. If u_1 is a non-negative classical solution of the equation $\Delta u + au = p(x)f(u)$ in Ω such that $u_1 \not\equiv 0$ on $\partial\Omega$ then u_1 is positive in Ω.*

Proof. Let $\Phi \in C^{0,\mu}(\partial\Omega)$ be such that $\Phi \not\equiv 0$ and $0 \leq \Phi \leq u_1$ on $\partial\Omega$. Consider the problem

$$\begin{cases} \Delta u = |a|u + \|p\|_\infty f(u) & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega. \end{cases} \tag{42}$$

By Theorem 4, this problem has a unique solution, say u_0 and, moreover, $u_0 > 0$ in Ω. But u_1 is supersolution for problem (42), so $u_1 \ge u_0 > 0$ in Ω and our claim is proved. ■

Proof of Theorem 4. We first observe that $u_- = 0$ is a sub-solution of (40), while $u^+ = n$ is a super-solution of (40) if n is large enough. Hence problem (40) has at least a solution u_Φ.

Then, taking into account the regularity of p and f, a standard boot-strap argument based on Schauder and Hölder regularity shows that $u_\Phi \in C^2(\Omega) \cap C(\overline{\Omega})$. The fact that u_Φ is the unique classical solution to (40) follows in the same way as in [11, Theorem 5].

We state in what follows two proofs for the positivity of u_Φ: the first one relies essentially on Theorem 1.20 in [14] while the second proof offers a more easier and direct approach.

First proof: Set $M := \max_{\overline{\Omega}} p$. Let u_* be the unique non-negative classical solution of the problem

$$\begin{cases} \Delta u_* = Mf(u_*) & \text{in } \Omega, \\ u_* = \Phi & \text{on } \partial\Omega. \end{cases}$$

To conclude that $u_\Phi > 0$ in Ω it is enough to show that $u_\Phi \ge u_* > 0$ in Ω. Since $f \in C^1[0,\infty)$ we have

$$\lim_{u\to 0^+} \frac{u^2}{F(u)} = \lim_{u\to 0^+} \frac{2u}{f(u)} = \frac{2}{f'(0)} > 0 \tag{43}$$

which implies immediately that $\int_{0^+}^1 \frac{du}{\sqrt{F(u)}} = \infty$. By applying Theorem 1.20 in Diaz [14], we conclude that $u_* > 0$ in Ω.

We now prove that $u_\Phi \ge u_*$ in Ω. To this aim, fix $\varepsilon > 0$. We claim that

$$u_*(x) \le u_\Phi(x) + \varepsilon(1+|x|^2)^{-1/2} \quad \text{for any } x \in \Omega. \tag{44}$$

Assume the contrary. Since $u_{*\,|\partial\Omega} = u_{\Phi\,|\partial\Omega} = \Phi$ we deduce that

$$\max_{x\in\overline{\Omega}} \{u_*(x) - u_\Phi(x) - \varepsilon(1+|x|^2)^{-1/2}\}$$

is achieved in Ω. At that point we have

$$\begin{aligned} 0 &\ge \Delta\left(u_*(x) - u_\Phi(x) - \varepsilon(1+|x|^2)^{-1/2}\right) = Mf(u_*(x)) - p(x)f(u_\Phi(x)) - \varepsilon\Delta(1+|x|^2)^{-1/2} \\ &\ge p(x)\left(f(u_*(x)) - f(u_\Phi(x))\right) + \varepsilon(N-3)(1+|x|^2)^{-3/2} + 3\,\varepsilon(1+|x|^2)^{-5/2} > 0\,, \end{aligned}$$

which is a contradiction. Since $\varepsilon > 0$ is chosen arbitrarily, inequality (44) implies $u_\Phi \ge u_*$ in Ω. ■

Second proof: Since $\Phi \not\equiv 0$, there exists $x_0 \in \Omega$ such that $u_\Phi(x_0) > 0$. To conclude that $u_\Phi > 0$ in Ω it is sufficient to prove that $u_\Phi > 0$ on $B(x_0;\overline{r})$ where $\overline{r} = \operatorname{dist}(x_0, \partial\Omega)$. Without loss of generality we can assume $x_0 = 0$. By the continuity of u_Φ, there exists $\underline{r} \in (0, \overline{r})$ such that $u_\Phi(x) > 0$ for all x with $|x| \le \underline{r}$. So, $\min_{|x|=\underline{r}} u_\Phi(x) =: \rho > 0$. We define

$$M := \max_{\overline{\Omega}} p, \quad \eta := \int_\rho^{\rho+1} \frac{dt}{f(t)} \quad \text{and} \quad \nu(\varepsilon) := \int_\varepsilon^{\rho+1} \frac{dt}{f(t)} \quad \text{for } 0 < \varepsilon < \rho.$$

It remains to show that $u_\Phi > 0$ in $A(\underline{r}, \overline{r})$, where

$$A(\underline{r}, \overline{r}) := \{x \in \mathbf{R}^N : \ \underline{r} < |x| < \overline{r}\}.$$

For this aim, we need the following lemma.

Lemma 13 *For $\varepsilon > 0$ small enough, the problem*

$$\begin{cases} -\Delta v = M & \text{in } A(\underline{r}, \overline{r}), \\ v(x) = \eta & \text{as } |x| = \underline{r}, \\ v(x) = \nu(\varepsilon) & \text{as } |x| = \overline{r}, \end{cases} \tag{45}$$

has a unique solution, which is increasing in $A(\underline{r}, \overline{r})$.

Proof. By the maximum principle, the problem (45) has a unique solution. Moreover, v is radially symmetric in $A(\underline{r}, \overline{r})$, namely $v(x) = v(r)$, $r = |x|$. The function v satisfies

$$v''(r) + \frac{N-1}{r} v'(r) = -M, \quad \underline{r} < r < \overline{r}.$$

Integrating twice this relation we find

$$v(r) = -\frac{M}{2N} r^2 - \frac{C_1}{N-2} r^{2-N} + C_2, \quad \underline{r} < r < \overline{r},$$

where C_1 and C_2 are real constants. The boundary conditions $v(\underline{r}) = \eta$ and $v(\overline{r}) = \nu(\varepsilon)$ imply

$$C_1 = \left(\nu(\varepsilon) - \eta + \frac{M}{2N}(\overline{r}^2 - \underline{r}^2) \right) \frac{N-2}{\underline{r}^{2-N} - \overline{r}^{2-N}}.$$

From (43) we deduce that $\nu(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Thus, taking $\varepsilon > 0$ sufficiently small, C_1 becomes large enough to ensure that $v'(r) > 0$ for all $r \in (\underline{r}, \overline{r})$. ■

Set $\varepsilon > 0$ sufficiently small such that the conclusion of Lemma 13 holds. Let $\underline{u}$ be the function defined implicitely as follows

$$\int_{\underline{u}(x)+\varepsilon}^{\rho+1} \frac{dt}{f(t)} = v(x) \quad \text{for all } x \in A(\underline{r}, \overline{r}). \tag{46}$$

It is easy to check that

$$\begin{cases} \Delta \underline{u} \ge M f(\underline{u} + \varepsilon) \ge p(x) f(\underline{u}) & \text{in } A(\underline{r}, \overline{r}), \\ \underline{u}(x) = \rho - \varepsilon < u_\Phi(x) & \text{as } |x| = \underline{r}, \\ \underline{u}(x) = 0 \le u_\Phi(x) & \text{as } |x| = \overline{r}. \end{cases}$$

Using the maximum principle (as in the proof of (44)) we deduce that $\underline{u} \le u_\Phi$ in $A(\underline{r}, \overline{r})$. By (46) and Lemma 13 we deduce that $\underline{u}$ decreases in $A(\underline{r}, \overline{r})$. Thus, $\underline{u} > 0$ in $A(\underline{r}, \overline{r})$. This completes the proof. ■

The positiveness of the solution in Theorem 4 follows essentially by the assumption $f \in C^1$ on $[0, \infty)$. We show in what follows that if f is not differentiable at the origin, then problem (40) has a unique solution that is not necessarily positive in Ω. However, in this case, the positiveness of the solution may depend on c and on the geometry of Ω. Indeed, let us consider the problem

$$\begin{cases} \Delta u = \sqrt{u} & \text{in } \Omega, \\ u \ge 0 & \text{in } \Omega, \\ u = c & \text{on } \partial\Omega, \end{cases} \tag{47}$$

where $c > 0$ is a constant.

In order to justify the uniqueness, let u_1, u_2 be two solutions of (47). It is sufficient to show that $u_1 \le u_2$ in Ω. Set $\omega = \{x \in \Omega;\ u_1(x) > u_2(x)\}$ and assume that $\omega \neq \emptyset$. Then $\Delta(u_1 - u_2) = \sqrt{u_1} - \sqrt{u_2} > 0$ in ω and $u_1 - u_2 = 0$ on $\partial\omega$. The maximum principle implies $u_1 - u_2 \le 0$ in ω which yields a contradiction.

The existence of a solution follows after observing that $u_- = 0$ (resp. $u_+ = c$) are sub-solution (resp. super-solution) for our problem.

The following example illustrates that in certain situations the unique solution of the problem (47) may vanish.

EXAMPLE 1. Set $\Omega = B(0,1) \subset \mathbf{R}^N$ and $w(x) = a|x|^4$. If $c \le \frac{1}{(4N+8)^2}$, let us choose a so that $c \le a \le \frac{1}{(4N+8)^2}$. It follows that

$$\begin{cases} \Delta w = (4N+8)a|x|^2 \le \sqrt{a}|x|^2 = \sqrt{w} & \text{in } \Omega\,, \\ w = a \ge c & \text{on } \partial\Omega. \end{cases}$$

This means that w is a super-solution of (47). Since $w(0) = 0$ then, necessarily, $u(0) = 0$.

The next example shows that in some cases, depending on c and on $\operatorname{diam}\Omega$, the unique solution of (47) is positive.

EXAMPLE 2. Suppose that Ω can be included in a ball $B(x_0, R)$ with $R \le R_c := 2\sqrt[4]{c}\sqrt{N+2}$. Define $w(x) = a|x - x_0|^4$, where a is chosen so that $\frac{\sqrt{c}}{R^2} \ge \sqrt{a} \ge \frac{1}{4N+8}$. Then w satisfies

$$\begin{cases} \Delta w = (4N+8)a|x-x_0|^2 \ge \sqrt{a}|x-x_0|^2 = \sqrt{w} & \text{in } \Omega\,, \\ w = a|x-x_0|^4 \le c & \text{on } \partial\Omega \end{cases}$$

which shows that w is a sub-solution of (47). We conclude that $u(x) \ge w(x) > 0$, for any $x \in \Omega\backslash\{x_0\}$.

If $\operatorname{diam}\Omega < 2R \le 2R_c$, there exist two points x_0 and x_1 such that Ω can be included in each of the balls $B(x_0, R)$ and $B(x_1, R)$. Using the previous conclusion we have

$$u(x) \ge a \max\{|x-x_0|^4, |x-x_1|^4\} \ge a \left|\frac{x_1 - x_0}{2}\right|^4 > 0.$$

Choosing $a = \frac{c}{R^4}$, $|x_1 - x_0| = 2R - \operatorname{diam}\Omega$ and $R = R_c$, we find

$$u(x) \ge \frac{c}{R^4}\left(\frac{2R - \operatorname{diam}\,\Omega}{2}\right)^4 = c\left(1 - \frac{\operatorname{diam}\,\Omega}{2R}\right)^4 > 0, \quad \forall x \in \Omega.$$

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Entire solutions blowing-up at infinity for semilinear elliptic systems

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Abstract

We consider the system $\Delta u = p(x)g(v)$, $\Delta v = q(x)f(u)$ in $\mathbf{R}^N$, where f, g are positive and non-decreasing functions on $(0, \infty)$ satisfying the Keller-Osserman condition and we establish the existence of positive solutions that blow-up at infinity.

1 Introduction and the main results

Consider the following semilinear elliptic system

$$\begin{cases} \Delta u = p(x)g(v) & \text{in } \mathbf{R}^N, \\ \Delta v = q(x)f(u) & \text{in } \mathbf{R}^N, \end{cases} \tag{1}$$

where $N \geq 3$ and $p, q \in C^{0,\alpha}_{\text{loc}}(\mathbf{R}^N)$ $(0 < \alpha < 1)$ are non-negative and radially symmetric functions. Throughout this paper we assume that $f, g \in C^{0,\beta}_{\text{loc}}[0, \infty)$ $(0 < \beta < 1)$ are positive and non-decreasing on $(0, \infty)$.

We are concerned here with the existence of positive *entire large solutions* of (1), that is positive classical solutions which satisfy $u(x) \to \infty$ and $v(x) \to \infty$ as $|x| \to \infty$. Set $\mathbf{R}^+ = (0, \infty)$ and define

$$\mathcal{G} = \{(a, b) \in \mathbf{R}^+ \times \mathbf{R}^+;\ (\exists) \text{ an entire radial solution of (1) so that } (u(0), v(0)) = (a, b)\}.$$

The case of pure powers in the non-linearities was treated by Lair and Shaker in [5]. They proved that $\mathcal{G} = \mathbf{R}^+ \times \mathbf{R}^+$ if $f(t) = t^\gamma$ and $g(t) = t^\theta$ for $t \geq 0$ with $0 < \gamma, \theta \leq 1$. Moreover, they established that all positive entire radial solutions of (1) are *large* provided that

$$\int_0^\infty tp(t)\,dt = \infty, \qquad \int_0^\infty tq(t)\,dt = \infty. \tag{2}$$

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If, in turn
$$\int_0^\infty tp(t)\,dt < \infty, \qquad \int_0^\infty tq(t)\,dt < \infty \tag{3}$$
then all positive entire radial solutions of (1) are *bounded*.

Our purpose is to generalize the above results to a larger class of systems. More precisely, we prove

Theorem 1 *Assume that*
$$\lim_{t\to\infty} \frac{g(cf(t))}{t} = 0 \quad \text{for all } c > 0. \tag{4}$$
Then $\mathcal{G} = \mathbf{R}^+ \times \mathbf{R}^+$. Moreover, the following hold:
i) If p and q satisfy (2), then all positive entire radial solutions of (1) are large.
ii) If p and q satisfy (3), then all positive entire radial solutions of (1) are bounded. Furthermore, if f, g are locally Lipschitz continuous on $(0,\infty)$ and (u,v), $(\tilde{u},\tilde{v})$ denote two positive entire radial solutions of (1), then there exists a positive constant C such that for all $r \in [0,\infty)$
$$\max\{|u(r)-\tilde{u}(r)|, |v(r)-\tilde{v}(r)|\} \le C \max\{|u(0)-\tilde{u}(0)|, |v(0)-\tilde{v}(0)|\}.$$

If f and g satisfy the stronger regularity $f, g \in C^1[0,\infty)$, then we drop the assumption (4) and require, in turn,

$(\mathbf{H_1})$ $f(0) = g(0) = 0$, $\liminf_{u\to\infty} \frac{f(u)}{g(u)} =: \sigma > 0$

and the Keller-Osserman condition (see [4, 10])

$(\mathbf{H_2})$ $\displaystyle\int_1^\infty \frac{dt}{\sqrt{G(t)}} < \infty$, where $G(t) = \displaystyle\int_0^t g(s)\,ds$.

Observe that assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ imply that f satisfies condition $(\mathbf{H_2})$, too.

The significance of the growth condition $(\mathbf{H_2})$ in the scalar case will be stated in the next Section.

Set $\eta = \min\{p,q\}$. If η is not identically zero at infinity and assumption (3) holds, then we prove

Property 1: $\mathcal{G} \neq \emptyset$ (see Lemma 4).

Property 2: $\mathcal{G}$ is *bounded* (see Lemma 5).

Property 3: $F(\mathcal{G}) \subset \mathcal{G}$ (see Lemma 6), where
$$F(\mathcal{G}) = \{(a,b) \in \partial\mathcal{G} \mid a > 0 \text{ and } b > 0\}.$$

For $(c,d) \in (\mathbf{R}^+ \times \mathbf{R}^+) \setminus \mathcal{G}$, define
$$R_{c,d} = \sup\{r > 0 \,|\, \text{there exists a radial solution of (1) in } B(0,r) \text{ so that } (u(0),v(0)) = (c,d)\}. \tag{5}$$

Property 4: $0 < R_{c,d} < \infty$ provided that $\nu = \max\{p(0), q(0)\} > 0$ (see Lemma 7).

Our main result in this case is

Theorem 2 *Let $f, g \in C^1[0,\infty)$ satisfy $(\mathbf{H_1})$ and $(\mathbf{H_2})$. Assume (3) holds, η is not identically zero at infinity and $\nu > 0$. Then any entire radial solution (u,v) of (1) with $(u(0),v(0)) \in F(\mathcal{G})$ is large.*

2 Preliminaries

Let $\Omega \subseteq \mathbf{R}^N$, $N \geq 3$ denote a smooth bounded domain or the whole space $\mathbf{R}^N$. Assume $\rho \not\equiv 0$ is non-negative such that $\rho \in C^{0,\alpha}(\overline{\Omega})$, if Ω is bounded and $\rho \in C^{0,\alpha}_{\text{loc}}(\Omega)$ otherwise. Consider the problem

$$\Delta u = \rho(x)h(u) \quad \text{in } \Omega, \tag{6}$$

where the non-linearity $h \in C^1[0,\infty)$ satisfies

$(\mathbf{A_1})$ $\quad h(0) = 0,\ h' \geq 0,\ h > 0 \quad \text{on } (0,\infty).$

Proposition 1 *Let $\Omega = B(0,R)$ for some $R > 0$ and let ρ be radially symmetric in Ω. Then Eq. (6) subject to the Dirichlet boundary condition*

$$u = c\ (const.) > 0 \quad on\ \partial\Omega, \tag{7}$$

has a unique non-negative solution u_c, which, moreover, is positive and radially symmetric.

Proof. By Proposition 2.1 in [8] (see also [1, Theorem 5]), problem (6)+(7) has a unique non-negative solution u_c which, moreover, is positive. If u_c were not radially symmetric, then a different solution could be obtained by rotating it, which would contradict the uniqueness of the solution. ■

By a *large solution* of Eq. (6) we mean a solution $u \geq 0$ in Ω satisfying $u(x) \to \infty$ as $\text{dist}\,(x, \partial\Omega) \to 0$ (if $\Omega \not\equiv \mathbf{R}^N$) or $u(x) \to \infty$ as $|x| \to \infty$ (if $\Omega = \mathbf{R}^N$). In the latter case, the solution is called an *entire large solution.* We point out that, if there exists a large solution of Eq. (6), then it is *positive.* Indeed, assume that $u(x_0) = 0$ for some $x_0 \in \Omega$. Since u is a large solution we can find a smooth domain $\omega \subset\subset \Omega$ such that $x_0 \in \omega$ and $u > 0$ on $\partial\omega$. Thus, by Theorem 5 in [1], the problem

$$\begin{cases} \Delta\zeta = \rho(x)h(\zeta) & \text{in } \omega, \\ \zeta = u & \text{on } \partial\omega, \\ \zeta \geq 0 & \text{in } \omega \end{cases}$$

has a unique solution, which is positive. By uniqueness, $\zeta = u$ in ω, which is a contradiction. This shows that any large solution of Eq. (6) cannot vanish in Ω.

Cf. Keller [4] and Osserman [10], if Ω is bounded and $\rho \equiv 1$, then Eq. (6) has a large solution if and only if h satisfies

$(\mathbf{A_2})$ $\quad \displaystyle\int_1^\infty \frac{dt}{\sqrt{H(t)}} < \infty, \quad \text{where } H(t) = \int_0^t h(s)\,ds.$

This fact leads to

Lemma 1 *Eq. (6), considered in bounded domains, can have large solutions only if h satisfies the Keller-Osserman condition $(\mathbf{A_2})$.*

Proof. Suppose, a priori, that Eq. (6) has a large solution u_∞. For any $n \geq 1$, consider the problem

$$\begin{cases} \Delta u = \|\rho\|_\infty h(u) & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

By Proposition 2.1 in [8], this problem has a unique solution, say u_n, which, moreover, is positive in $\overline{\Omega}$. By the maximum principle

$$0 < u_n \leq u_{n+1} \leq u_\infty \quad \text{in } \Omega, \quad \forall n \geq 1.$$

Thus, for every $x \in \Omega$, it makes sense to define $\overline{u}(x) = \lim_{n\to\infty} u_n(x)$. Since (u_n) is uniformly bounded on every compact set $\omega \subset\subset \Omega$, standard elliptic regularity implies that $\overline{u}$ is a large solution of the problem $\Delta u = \|\rho\|_\infty h(u)$ in Ω. ■

Therefore, in the rest of this section, we consider Eq. (6) assuming always that $(\mathbf{A_1})$ and $(\mathbf{A_2})$ hold. In this situation, by Lemma 1 in [1],

$$\int_1^\infty \frac{dt}{h(t)} < \infty. \tag{8}$$

Typical examples of non-linearities satisfying $(\mathbf{A_1})$ and $(\mathbf{A_2})$ are: i) $h(u) = e^u - 1$; ii) $h(u) = u^p$, $p > 1$; iii) $h(u) = u[\ln(u+1)]^p$, $p > 2$.

For the proofs of the Propositions that will be stated below, we refer the reader to [1].

Proposition 2 ([1, Theorem 1].) *Let Ω be a bounded domain. Assume that ρ satisfies*

(ρ_1) *for every $x_0 \in \Omega$ with $\rho(x_0) = 0$, there is a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega}_0 \subset \Omega$ and $\rho_{|\partial\Omega_0} > 0$.*

Then Eq. (6) possesses a large solution.

Corollary 1 *Let $\Omega = B(0,R)$ for some $R > 0$. If ρ is radially symmetric in Ω and $\rho_{|\partial\Omega} > 0$, then there exists a radial large solution of Eq. (6).*

Proof. By Proposition 1, the large solution constructed in the same way as in the proof of [1, Theorem 1] will be radially symmetric. ■

Proposition 3 ([1, Theorem 2].) *Consider Eq. (6) with $\Omega = \mathbf{R}^N$ assuming that ρ satisfies*

$(\rho_1)'$ *There exists a sequence of smooth bounded domains $(\Omega_n)_{n\geq 1}$ such that $\overline{\Omega}_n \subset \Omega_{n+1}$, $\mathbf{R}^N = \cup_{n=1}^\infty \Omega_n$ and (ρ_1) holds in Ω_n, for any $n \geq 1$.*

(ρ_2) $\displaystyle\int_0^\infty r\varphi(r)\,dr < \infty,$ *where $\varphi(r) = \max\{\rho(x) : |x| = r\}$.*

Then Eq. (6) has an entire large solution.

Remark 1 *Theorem 4 in [1] asserts that (8) is a necessary condition for the existence of entire large solutions to Eq. (6) if ρ satisfies (ρ_2) and for which h is not assumed to fulfill $(\mathbf{A_2})$.*

Remark 2 *If ρ is radially symmetric in $\mathbf{R}^N$ and not identically zero at infinity, then $(\rho_1)'$ is fulfilled.*

Indeed, we can find an increasing sequence of positive numbers $(R_n)_{n\geq 1}$ such that $R_n \to \infty$ and $\rho > 0$ on $\partial B(0, R_n)$, for any $n \geq 1$. Therefore, $(\rho_1)'$ is satisfied on $\Omega_n = B(0, R_n)$.

Corollary 2 *Let $\Omega \equiv \mathbf{R}^N$. Assume that ρ is radially symmetric in $\mathbf{R}^N$, not identically zero at infinity such that (ρ_2) is fulfilled. Then Eq. (6) has a radial entire large solution.*

Proof. By Remark 2 and Corollary 1, the entire large solution constructed as in the proof of Theorem 2 in [1] will be radially symmetric. ∎

We supplied in [1] an example of function ρ with properties stated in Corollary 2. More precisely,

$$\begin{cases} \rho(r) = 0 \quad \text{for} \quad r = |x| \in [n - 1/3, n + 1/3], \ n \geq 1; \\ \rho(r) > 0 \quad \text{in} \quad \mathbf{R}_+ \setminus \bigcup_{n=1}^{\infty} [n - 1/3, n + 1/3]; \\ \rho \in C^1[0, \infty) \quad \text{and} \quad \max_{r \in [n, n+1]} \rho(r) = \dfrac{1}{n^3}. \end{cases}$$

3 Auxiliary results

Lemma 2 *Condition (2) holds if and only if $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$ where*

$$A(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt, \qquad B(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} q(s)\, ds\, dt, \qquad \forall r > 0.$$

Proof. Indeed, for any $r > 0$

$$A(r) = \frac{1}{N-2} \left[\int_0^r tp(t)\, dt - \frac{1}{r^{N-2}} \int_0^r t^{N-1} p(t)\, dt \right] \leq \frac{1}{N-2} \int_0^r tp(t)\, dt. \tag{9}$$

On the other hand,

$$\begin{aligned} \int_0^r tp(t)\, dt - \frac{1}{r^{N-2}} \int_0^r t^{N-1} p(t)\, dt &= \frac{1}{r^{N-2}} \int_0^r \left(r^{N-2} - t^{N-2} \right) tp(t)\, dt \\ &\geq \frac{1}{r^{N-2}} \left[r^{N-2} - \left(\frac{r}{2} \right)^{N-2} \right] \int_0^{\frac{r}{2}} tp(t)\, dt. \end{aligned}$$

This combined with (9) yields

$$\frac{1}{N-2} \int_0^r tp(t)\, dt \geq A(r) \geq \frac{1}{N-2} \left[1 - \left(\frac{1}{2} \right)^{N-2} \right] \int_0^{\frac{r}{2}} tp(t)\, dt.$$

Our conclusion follows now by letting $r \to \infty$. ∎

Lemma 3 *Assume that condition (3) holds. Let f and g be locally Lipschitz continuous functions on $(0,\infty)$. If (u,v) and $(\tilde{u},\tilde{v})$ denote two bounded positive entire radial solutions of (1), then there exists a positive constant C such that for all $r\in[0,\infty)$*

$$\max\{|u(r)-\tilde{u}(r)|,|v(r)-\tilde{v}(r)|\}\le C\max\{|u(0)-\tilde{u}(0)|,|v(0)-\tilde{v}(0)|\}.$$

Proof. We first see that radial solutions of (1) are solutions of the ordinary differential equations system

$$\begin{cases} u''(r)+\dfrac{N-1}{r}u'(r)=p(r)\,g(v(r)), & r>0\\ v''(r)+\dfrac{N-1}{r}v'(r)=q(r)\,f(u(r)), & r>0.\end{cases}\tag{10}$$

Define $K=\max\{|u(0)-\tilde{u}(0)|,|v(0)-\tilde{v}(0)|\}$. Integrating the first equation of (10), we get

$$u'(r)-\tilde{u}'(r)=r^{1-N}\int_0^r s^{N-1}p(s)(g(v(s))-g(\tilde{v}(s)))\,ds.$$

Hence

$$|u(r)-\tilde{u}(r)|\le K+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)|g(v(s))-g(\tilde{v}(s))|\,ds\,dt.\tag{11}$$

Since (u,v) and $(\tilde{u},\tilde{v})$ are bounded entire radial solutions of (1) we have

$$\begin{aligned} &|g(v(r))-g(\tilde{v}(r))|\le m|v(r)-\tilde{v}(r)| && \text{for any } r\in[0,\infty)\\ &|f(u(r))-f(\tilde{u}(r))|\le m|u(r)-\tilde{u}(r)| && \text{for any } r\in[0,\infty),\end{aligned}$$

where m denotes a Lipschitz constant for both functions f and g. Therefore, using (11) we find

$$|u(r)-\tilde{u}(r)|\le K+m\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)|v(s)-\tilde{v}(s)|\,ds\,dt.\tag{12}$$

Arguing as above, but now with the second equation of (10), we obtain

$$|v(r)-\tilde{v}(r)|\le K+m\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)|u(s)-\tilde{u}(s)|\,ds\,dt.\tag{13}$$

Define

$$X(r)=K+m\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)|v(s)-\tilde{v}(s)|\,ds\,dt.$$

$$Y(r)=K+m\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)|u(s)-\tilde{u}(s)|\,ds\,dt.$$

It is clear that X and Y are non-decreasing functions with $X(0)=Y(0)=K$. By a simple calculation together with (12) and (13) we obtain

$$\begin{aligned}(r^{N-1}X')'(r)&=mr^{N-1}p(r)|v(r)-\tilde{v}(r)|\le mr^{N-1}p(r)Y(r)\\ (r^{N-1}Y')'(r)&=mr^{N-1}q(r)|u(r)-\tilde{u}(r)|\le mr^{N-1}q(r)X(r).\end{aligned}\tag{14}$$

Since Y is non-decreasing, we have

$$X(r) \leq K + mY(r)A(r) \leq K + \frac{m}{N-2} Y(r) \int_0^r tp(t)\,dt \leq K + mC_pY(r) \qquad (15)$$

where $C_p = (1/(N-2)) \int_0^\infty tp(t)\,dt$. Using (15) in the second inequality of (14) we find

$$(r^{N-1}Y')'(r) \leq mr^{N-1}q(r)(K + mC_pY(r)).$$

Integrating twice this inequality from 0 to r, we obtain

$$Y(r) \leq K(1 + mC_q) + \frac{m^2}{N-2} C_p \int_0^r tq(t)Y(t)\,dt,$$

where $C_q = (1/(N-2)) \int_0^\infty tq(t)\,dt$. From Gronwall's inequality, we deduce

$$Y(r) \leq K(1 + mC_q)e^{\frac{m^2}{N-2}C_p \int_0^r tq(t)\,dt} \leq K(1 + mC_q)e^{m^2C_pC_q}$$

and similarly for X. The conclusion follows now from the above inequality, (12) and (13). ■

4 Proof of Theorem 1

Since the radial solutions of (1) are solutions of the ordinary differential equations system (10) it follows that the radial solutions of (1) with $u(0) = a > 0$, $v(0) = b > 0$ satisfy

$$u(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1}p(s)\,g(v(s))\,ds\,dt, \qquad r \geq 0. \qquad (16)$$

$$v(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1}q(s)\,f(u(s))\,ds\,dt, \qquad r \geq 0. \qquad (17)$$

Define $v_0(r) = b$ for all $r \geq 0$. Let $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ be two sequences of functions given by

$$u_k(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1}p(s)\,g(v_{k-1}(s))\,ds\,dt, \qquad r \geq 0.$$

$$v_k(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1}q(s)\,f(u_k(s))\,ds\,dt, \qquad r \geq 0.$$

Since $v_1(r) \geq b$, we find $u_2(r) \geq u_1(r)$ for all $r \geq 0$. This implies $v_2(r) \geq v_1(r)$ which further produces $u_3(r) \geq u_2(r)$ for all $r \geq 0$. Proceeding at the same manner we conclude that

$$u_k(r) \leq u_{k+1}(r) \quad \text{and} \quad v_k(r) \leq v_{k+1}(r), \qquad \forall r \geq 0 \text{ and } k \geq 1.$$

We now prove that the non-decreasing sequences $(u_k(r))_{k\geq 1}$ and $(v_k(r))_{k\geq 1}$ are bounded from above on bounded sets. Indeed, we have

$$u_k(r) \leq u_{k+1}(r) \leq a + g(v_k(r))A(r), \qquad \forall r \geq 0 \qquad (18)$$

and

$$v_k(r) \leq b + f\,(u_k(r))\,B(r), \qquad \forall r \geq 0. \qquad (19)$$

Let $R>0$ be arbitrary. By (18) and (19) we find

$$u_k(R) \le a + g\left(b + f\left(u_k(R)\right) B(R)\right) A(R), \qquad \forall k \ge 1$$

or, equivalently,

$$1 \le \frac{a}{u_k(R)} + \frac{g\left(b + f\left(u_k(R)\right) B(R)\right)}{u_k(R)} A(R), \qquad \forall k \ge 1. \tag{20}$$

By the monotonicity of $(u_k(R))_{k\ge1}$, there exists $\lim_{k\to\infty} u_k(R) := L(R)$. We claim that $L(R)$ is finite. Assume the contrary. Then, by taking $k \to \infty$ in (20) and using (4) we obtain a contradiction. Since $u_k'(r),\ v_k'(r) \ge 0$ we get that the map $(0,\infty) \ni R \to L(R)$ is non-decreasing on $(0,\infty)$ and

$$u_k(r) \le u_k(R) \le L(R), \qquad \forall r \in [0,R],\ \forall k \ge 1. \tag{21}$$

$$v_k(r) \le b + f\left(L(R)\right) B(R), \qquad \forall r \in [0,R],\ \forall k \ge 1. \tag{22}$$

It follows that there exists $\lim_{R\to\infty} L(R) = \overline{L} \in (0,\infty]$ and the sequences $(u_k(r))_{k\ge1}$, $(v_k(r))_{k\ge1}$ are bounded above on bounded sets. Therefore, we can define $u(r) := \lim_{k\to\infty} u_k(r)$ and $v(r) := \lim_{k\to\infty} v_k(r)$ for all $r \ge 0$. By standard elliptic regularity theory we obtain that (u,v) is a positive entire solution of (1) with $u(0)=a$ and $v(0)=b$.

We now assume that, in addition, condition (3) is fulfilled. According to Lemma 2 we have that $\lim_{r\to\infty} A(r) = \overline{A} < \infty$ and $\lim_{r\to\infty} B(r) = \overline{B} < \infty$. Passing to the limit as $k\to\infty$ in (20) we find

$$1 \le \frac{a}{L(R)} + \frac{g\left(b + f\left(L(R)\right) B(R)\right)}{L(R)} A(R) \le \frac{a}{L(R)} + \frac{g(b + f\left(L(R)\right)\overline{B})}{L(R)} \overline{A}.$$

Letting $R\to\infty$ and using (4) we deduce $\overline{L} < \infty$. Thus, taking into account (21) and (22), we obtain

$$u_k(r) \le \overline{L} \quad \text{and} \quad v_k(r) \le b + f(\overline{L})\overline{B}, \qquad \forall r \ge 0,\ \forall k \ge 1.$$

So, we have found upper bounds for $(u_k(r))_{k\ge1}$ and $(v_k(r))_{k\ge1}$ which are independent of r. Thus, the solution (u,v) is bounded from above. This shows that any solution of (16) and (17) will be bounded from above provided (3) holds. Thus, we can apply Lemma 3 to achieve the second assertion of *ii*).

Let us now drop the condition (3) and assume that (2) is fulfilled. In this case, Lemma 2 tells us that $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$. Let (u,v) be an entire positive radial solution of (1). Using (16) and (17) we obtain

$$u(r) \ge a + g(b)\, A(r), \qquad \forall r \ge 0.$$

$$v(r) \ge b + f(a)\, B(r), \qquad \forall r \ge 0.$$

Taking $r\to\infty$ we get that (u,v) is an entire large solution. This concludes the proof of Theorem 1. ∎

We now give some examples of non-linearities f and g which satisfy the assumptions of Theorem 1 (see [3]).

1) Let

$$f(t)=\sum_{j=1}^{l} a_j t^{\gamma_j}, \quad g(t)=\sum_{k=1}^{m} b_k t^{\theta_j} \quad \text{for } t>0$$

with a_j, b_k, γ_j, $\theta_k>0$ and $f(t)=g(t)=0$ for $t\le 0$. Assume that $\gamma\theta<1$, where

$$\gamma=\max_{1\le j\le l}\gamma_j, \quad \theta=\max_{1\le k\le m}\theta_k.$$

2) Let

$$f(t)=(1+t^2)^{\gamma/2} \quad \text{and} \quad g(t)=(1+t^2)^{\theta/2} \quad \text{for } t\in\mathbf{R}$$

with γ, $\theta>0$ and $\gamma\theta<1$.

3) Let

$$f(t)=\begin{cases} t^{\gamma} & \text{if } 0\le t\le 1,\\ t^{\theta} & \text{if } t\ge 1,\end{cases}$$

and

$$g(t)=\begin{cases} t^{\theta} & \text{if } 0\le t\le 1,\\ t^{\gamma} & \text{if } t\ge 1,\end{cases}$$

with γ, $\theta>0$, $\gamma\theta<1$ and $f(t)=g(t)=0$ for $t\le 0$.

4) Let $g(t)=t$ for $t\in\mathbf{R}$, $f(t)=0$ for $t\le 0$ and

$$f(t)=t\left(-\ln\left(\left(\frac{2}{\pi}\right)\arctan t\right)\right)^{\gamma} \quad \text{for } t>0$$

where $\gamma\in(0,1/2)$.

5 Proof of Theorem 2

Let $f,g\in C^1[0,\infty)$ satisfy $(\mathbf{H_1})$ and $(\mathbf{H_2})$. Suppose that η is not identically zero at infinity and (3) holds. We first give the proofs of Properties 1-4 which are the main tools used to deduce Theorem 2.

Lemma 4 $\mathcal{G}\neq\emptyset$.

Proof. By Corollary 2, the problem

$$\Delta\psi=(p+q)(x)(f+g)(\psi) \quad \text{in } \mathbf{R}^N,$$

has a positive radial entire large solution. Since ψ is radial, we have

$$\psi(r)=\psi(0)+\int_0^r t^{1-N}\int_0^t s^{N-1}(p+q)(s)(f+g)(\psi(s))\,ds\,dt, \quad \forall r\ge 0.$$

We claim that $(0,\psi(0)]\times(0,\psi(0)]\subseteq\mathcal{G}$. To prove this, fix $0<a,b\le\psi(0)$ and let $v_0(r)\equiv b$ for all $r\ge 0$. Define the sequences $(u_k)_{k\ge1}$ and $(v_k)_{k\ge1}$ by

$$u_k(r)=a+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g(v_{k-1}(s))\,ds\,dt,\quad \forall r\in[0,\infty),\quad \forall k\ge 1, \tag{23}$$

$$v_k(r)=b+\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f(u_k(s))\,ds\,dt,\quad \forall r\in[0,\infty),\quad \forall k\ge 1. \tag{24}$$

We first see that $v_0\le v_1$ which produces $u_1\le u_2$. Consequently, $v_1\le v_2$ which further yields $u_2\le u_3$. With the same arguments, we obtain that (u_k) and (v_k) are non-decreasing sequences. Since $\psi'(r)\ge 0$ and $b=v_0\le\psi(0)\le\psi(r)$ for all $r\ge 0$ we find

$$\begin{aligned}u_1(r)\ &\le a+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g(\psi(s))\,ds\,dt\\ &\le \psi(0)+\int_0^r t^{1-N}\int_0^t s^{N-1}(p+q)(s)(f+g)(\psi(s))\,ds\,dt=\psi(r).\end{aligned}$$

Thus $u_1\le\psi$. It follows that

$$\begin{aligned}v_1(r)\ &\le b+\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f(\psi(s))\,ds\,dt\\ &\le \psi(0)+\int_0^r t^{1-N}\int_0^t s^{N-1}(p+q)(s)(f+g)(\psi(s))\,ds\,dt=\psi(r).\end{aligned}$$

Similar arguments show that

$$u_k(r)\le\psi(r)\quad\text{and}\quad v_k(r)\le\psi(r)\quad \forall r\in[0,\infty),\quad \forall k\ge 1.$$

Thus, (u_k) and (v_k) converge and $(u,v)=\lim_{k\to\infty}(u_k,v_k)$ is an entire radial solution of (1) such that $(u(0),v(0))=(a,b)$. This completes the proof. ∎

An easy consequence of the above result is

Corollary 3 *If $(a,b)\in\mathcal{G}$, then $(0,a]\times(0,b]\subseteq\mathcal{G}$.*

Proof. Indeed, the process used before can be repeated by taking

$$u_k(r)=a_0+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g(v_{k-1}(s))\,ds\,dt,\quad \forall r\in[0,\infty),\quad \forall k\ge 1,$$

$$v_k(r)=b_0+\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f(u_k(s))\,ds\,dt,\quad \forall r\in[0,\infty),\quad \forall k\ge 1,$$

where $0<a_0\le a$, $0<b_0\le b$ and $v_0(r)\equiv b_0$ for all $r\ge 0$.

Letting (U,V) be the entire radial solution of (1) with central values (a,b) we obtain as in Lemma 4,

$$u_k(r)\le u_{k+1}(r)\le U(r),\quad \forall r\in[0,\infty),\quad \forall k\ge 1,$$

$$v_k(r)\le v_{k+1}(r)\le V(r),\quad \forall r\in[0,\infty),\quad \forall k\ge 1.$$

Set $(u,v)=\lim_{k\to\infty}(u_k,v_k)$. We see that $u\le U$, $v\le V$ on $[0,\infty)$ and (u,v) is an entire radial solution of (1) with central values (a_0,b_0). This shows that $(a_0,b_0)\in\mathcal{G}$, so that our assertion is proved. ∎

Lemma 5 *$\mathcal{G}$ is bounded.*

Proof. Set $0 < \lambda < \min\{\sigma, 1\}$ and let $\delta = \delta(\lambda)$ be large enough so that

$$f(t) \geq \lambda g(t), \qquad \forall t \geq \delta. \tag{25}$$

Since η is radially symmetric and not identically zero at infinity, we can assume $\eta > 0$ on $\partial B(0, R)$ for some $R > 0$. Corollary 1 ensures the existence of a positive large solution ζ of the problem

$$\Delta\zeta = \lambda\eta(x)g\left(\frac{\zeta}{2}\right) \qquad \text{in } B(0, R).$$

Arguing by contradiction, let us assume that $\mathcal{G}$ is not bounded. Then, there exists $(a, b) \in \mathcal{G}$ such that $a + b > \max\{2\delta, \zeta(0)\}$. Let (u, v) be the entire radial solution of (1) such that $(u(0), v(0)) = (a, b)$. Since $u(x) + v(x) \geq a + b > 2\delta$ for all $x \in \mathbf{R}^N$, by (25), we find

$$f(u(x)) \geq f\left(\frac{u(x) + v(x)}{2}\right) \geq \lambda g\left(\frac{u(x) + v(x)}{2}\right) \qquad \text{if } u(x) \geq v(x)$$

and

$$g(v(x)) \geq g\left(\frac{u(x) + v(x)}{2}\right) \geq \lambda g\left(\frac{u(x) + v(x)}{2}\right) \qquad \text{if } v(x) \geq u(x).$$

It follows that

$$\Delta(u + v) = p(x)g(v) + q(x)f(u) \geq \eta(x)(g(v) + f(u)) \geq \lambda\eta(x)g\left(\frac{u + v}{2}\right) \qquad \text{in } \mathbf{R}^N.$$

On the other hand, $\zeta(x) \to \infty$ as $|x| \to R$ and $u, v \in C^2(\overline{B(0, R)})$. Thus, by the maximum principle, we conclude that $u + v \leq \zeta$ in $B(0, R)$. But this is impossible since $u(0) + v(0) = a + b > \zeta(0)$. ∎

Lemma 6 $F(\mathcal{G}) \subset \mathcal{G}$.

Proof. Let $(a, b) \in F(\mathcal{G})$. We claim that $(a - 1/n_0, b - 1/n_0) \in \mathcal{G}$ provided $n_0 \geq 1$ is large enough so that $\min\{a, b\} > 1/n_0$. Indeed, if this is not true, by Corollary 3

$$D := \left[a - \frac{1}{n_0}, \infty\right) \times \left[b - \frac{1}{n_0}, \infty\right) \subseteq (\mathbf{R}^+ \times \mathbf{R}^+) \setminus \mathcal{G}.$$

So, we can find a small ball B centered in (a, b) such that $B \subset\subset D$, i.e., $B \cap \mathcal{G} = \emptyset$. But this will contradict the choice of (a, b). Consequently, there exists (u_{n_0}, v_{n_0}) an entire radial solution of (1) such that $(u_{n_0}(0), v_{n_0}(0)) = (a - 1/n_0, b - 1/n_0)$. Thus, for any $n \geq n_0$, we can define

$$u_n(r) = a - \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(v_n(s))\, ds\, dt, \qquad r \geq 0,$$

$$v_n(r) = b - \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(u_n(s))\, ds\, dt, \qquad r \geq 0.$$

Using Corollary 3 once more, we conclude that $(u_n)_{n \geq n_0}$ and $(v_n)_{n \geq n_0}$ are non-decreasing sequences. We now prove that (u_n) and (v_n) converge on $\mathbf{R}^N$. To this aim, let $x_0 \in \mathbf{R}^N$ be arbitrary. But η is not identically zero at infinity so that, for some $R_0 > 0$, we have $\eta > 0$ on $\partial B(0, R_0)$ and $x_0 \in B(0, R_0)$.

Since $\sigma = \liminf_{u\to\infty} \frac{f(u)}{g(u)} > 0$, we find $\tau \in (0, 1)$ such that

$$f(t) \geq \tau g(t), \qquad \forall t \geq \frac{a+b}{2} - \frac{1}{n_0}.$$

Therefore, on the set where $u_n \geq v_n$, we have

$$f(u_n) \geq f\left(\frac{u_n + v_n}{2}\right) \geq \tau g\left(\frac{u_n + v_n}{2}\right).$$

Similarly, on the set where $u_n \leq v_n$, we have

$$g(v_n) \geq g\left(\frac{u_n + v_n}{2}\right) \geq \tau g\left(\frac{u_n + v_n}{2}\right).$$

It follows that, for any $x \in \mathbf{R}^N$,

$$\Delta(u_n + v_n) = p(x)g(v_n) + q(x)f(u_n) \geq \eta(x)[g(v_n) + f(u_n)] \geq \tau\eta(x)g\left(\frac{u_n + v_n}{2}\right).$$

On the other hand, by Corollary 1, there exists a positive large solution of

$$\Delta\zeta = \tau\eta(x)g\left(\frac{\zeta}{2}\right) \qquad \text{in } B(0, R_0).$$

The maximum principle yields $u_n + v_n \leq \zeta$ in $B(0, R_0)$. So, it makes sense to define $(u(x_0), v(x_0)) = \lim_{n\to\infty}(u_n(x_0), v_n(x_0))$. Since x_0 is arbitrary, the functions u, v exist on $\mathbf{R}^N$. Hence (u, v) is an entire radial solution of (1) with central values (a, b), i.e., $(a, b) \in \mathcal{G}$. ■

Lemma 7 *If, in addition,* $\nu = \max\{p(0), q(0)\} > 0$, *then* $0 < R_{c,d} < \infty$ *where* $R_{c,d}$ *is defined by (5).*

Proof. Since $\nu > 0$ and $p, q \in C[0, \infty)$, there exists $\epsilon > 0$ such that $(p + q)(r) > 0$ for all $0 \leq r < \epsilon$. Let $0 < R < \epsilon$ be arbitrary. By Corollary 1, there exists a positive radial large solution of the problem

$$\Delta\psi_R = (p + q)(x)(f + g)(\psi_R) \qquad \text{in } B(0, R).$$

Moreover, for any $0 \leq r < R$,

$$\psi_R(r) = \psi_R(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p + q)(s)(f + g)(\psi_R(s))\, ds\, dt.$$

It is clear that $\psi_R'(r) \geq 0$. Thus, we find

$$\psi_R'(r) = r^{1-N} \int_0^r s^{N-1}(p + q)(s)(f + g)(\psi_R(s))\, ds \leq C(f + g)(\psi_R(r))$$

where $C > 0$ is a positive constant such that $\int_0^\epsilon (p + q)(s)\, ds \leq C$.

Since $f+g$ satisfies $(\mathbf{A_1})$ and $(\mathbf{A_2})$, we may then invoke Lemma 1 in [1] to conclude

$$\int_1^\infty \frac{dt}{(f+g)(t)} < \infty.$$

Therefore, we get

$$-\frac{d}{dr}\int_{\psi_R(r)}^\infty \frac{ds}{(f+g)(s)} = \frac{\psi_R'(r)}{(f+g)(\psi_R(r))} \le C \qquad \text{for any } 0<r<R.$$

Integrating from 0 to R and recalling that $\psi_R(r)\to\infty$ as $r\nearrow R$, we obtain

$$\int_{\psi_R(0)}^\infty \frac{ds}{(f+g)(s)} \le CR.$$

Letting $R\searrow 0$ we conclude that

$$\lim_{R\searrow 0}\int_{\psi_R(0)}^\infty \frac{ds}{(f+g)(s)} = 0.$$

This implies that $\psi_R(0)\to\infty$ as $R\searrow 0$. So, there exists $0<\tilde{R}<\epsilon$ such that $0<c,d\le \psi_{\tilde{R}}(0)$. Set

$$u_k(r) = c + \int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g(v_{k-1}(s))\,ds\,dt, \qquad \forall r\in[0,\infty),\ \forall k\ge 1, \tag{26}$$

$$v_k(r) = d + \int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f(u_k(s))\,ds\,dt, \qquad \forall r\in[0,\infty),\ \forall k\ge 1, \tag{27}$$

where $v_0(r)=d$ for all $r\in[0,\infty)$. As in Lemma 4, we find that (u_k) resp., (v_k) are non-decreasing and

$$u_k(r)\le \psi_{\tilde{R}}(r) \quad \text{and} \qquad v_k(r)\le\psi_{\tilde{R}}(r), \qquad \forall r\in[0,\tilde{R}),\ \forall k\ge 1.$$

Thus, for any $r\in[0,\tilde{R})$, there exists $(u(r),v(r)) = \lim_{k\to\infty}(u_k(r),v_k(r))$ which is, moreover, a radial solution of (1) in $B(0,\tilde{R})$ such that $(u(0),v(0))=(c,d)$. This shows that $R_{c,d}\ge\tilde{R}>0$. By the definition of $R_{c,d}$ we also derive

$$\lim_{r\nearrow R_{c,d}} u(r)=\infty \quad \text{and} \qquad \lim_{r\nearrow R_{c,d}} v(r)=\infty. \tag{28}$$

On the other hand, since $(c,d)\notin\mathcal{G}$, we conclude that $R_{c,d}$ is finite. ■

Proof of Theorem 2 completed.

Let $(a,b)\in F(\mathcal{G})$ be arbitrary. By Lemma 6, $(a,b)\in\mathcal{G}$ so that we can define (U,V) an entire radial solution of (1) with $(U(0),V(0))=(a,b)$. Obviously, for any $n\ge 1$, $(a+1/n,b+1/n)\in(\mathbf{R}^+\times\mathbf{R}^+)\setminus\mathcal{G}$. By Lemma 7, $R_{a+1/n,b+1/n}$ (in short, R_n) defined by (5) is a positive number. Let (U_n,V_n) be the radial solution of (1) in $B(0,R_n)$ with the central values $(a+1/n,b+1/n)$. Thus,

$$U_n(r) = a+\frac{1}{n}+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g(V_n(s))\,ds\,dt, \qquad \forall r\in[0,R_n), \tag{29}$$

$$V_n(r) = b + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(U_n(s))\, ds\, dt, \qquad \forall r \in [0, R_n). \tag{30}$$

In view of (28) we have

$$\lim_{r \nearrow R_n} U_n(r) = \infty \quad \text{and} \quad \lim_{r \nearrow R_n} V_n(r) = \infty, \quad \forall n \geq 1.$$

We claim that $(R_n)_{n \geq 1}$ is a non-decreasing sequence. Indeed, if (u_k), (v_k) denote the sequences of functions defined by (26) and (27) with $c = a + 1/(n+1)$ and $d = b + 1/(n+1)$, then

$$u_k(r) \leq u_{k+1}(r) \leq U_n(r), \qquad v_k(r) \leq v_{k+1}(r) \leq V_n(r), \qquad \forall r \in [0, R_n),\ \forall k \geq 1. \tag{31}$$

This implies that $(u_k(r))_{k \geq 1}$ and $(v_k(r))_{k \geq 1}$ converge for any $r \in [0, R_n)$. Moreover, $(U_{n+1}, V_{n+1}) = \lim_{k \to \infty}(u_k, v_k)$ is a radial solution of (1) in $B(0, R_n)$ with central values $(a + 1/(n+1), b + 1/(n+1))$. By the definition of R_{n+1}, it follows that $R_{n+1} \geq R_n$ for any $n \geq 1$.

Set $R := \lim_{n \to \infty} R_n$ and let $0 \leq r < R$ be arbitrary. Then, there exists $n_1 = n_1(r)$ such that $r < R_n$ for all $n \geq n_1$. From (31) we see that $U_{n+1} \leq U_n$ (resp., $V_{n+1} \leq V_n$) on $[0, R_n)$ for all $n \geq 1$. So, there exists $\lim_{n \to \infty}(U_n(r), V_n(r))$ which, by (29) and (30), is a radial solution of (1) in $B(0, R)$ with central values (a, b). Consequently,

$$\lim_{n \to \infty} U_n(r) = U(r) \quad \text{and} \quad \lim_{n \to \infty} V_n(r) = V(r) \quad \text{for any } r \in [0, R). \tag{32}$$

Since $U_n'(r) \geq 0$, from (30) we find

$$V_n(r) \leq b + \frac{1}{n} + f(U_n(r)) \int_0^\infty t^{1-N} \int_0^t s^{N-1} q(s)\, ds\, dt.$$

This yields

$$V_n(r) \leq C_1 U_n(r) + C_2 f(U_n(r)) \tag{33}$$

where C_1 is an upper bound of $(V(0) + 1/n)/(U(0) + 1/n)$ and

$$C_2 = \int_0^\infty t^{1-N} \int_0^t s^{N-1} q(s)\, ds\, dt \leq \frac{1}{N-2} \int_0^\infty s q(s)\, ds < \infty.$$

Define $h(t) = g(C_1 t + C_2 f(t))$ for $t \geq 0$. It is easy to check that h satisfies $(\mathbf{A_1})$ and $(\mathbf{A_2})$. So, by Lemma 1 in [1] we can define

$$\Gamma(s) = \int_s^\infty \frac{dt}{h(t)}, \qquad \text{for all } s > 0.$$

But U_n verifies

$$\Delta U_n = p(x) g(V_n)$$

which combined with (33) implies

$$\Delta U_n \leq p(x) h(U_n).$$

A simple calculation shows that

$$\begin{aligned}\Delta\Gamma(U_n) &= \Gamma'(U_n)\Delta U_n + \Gamma''(U_n)|\nabla U_n|^2 = \frac{-1}{h(U_n)}\Delta U_n + \frac{h'(U_n)}{[h(U_n)]^2}|\nabla U_n|^2 \\ &\geq \frac{-1}{h(U_n)}p(r)h(U_n) = -p(r)\end{aligned}$$

which we rewrite as

$$\left(r^{N-1}\frac{d}{dr}\Gamma(U_n)\right)' \geq -r^{N-1}p(r) \qquad \text{for any } 0 < r < R_n.$$

Fix $0 < r < R$. Then $r < R_n$ for all $n \geq n_1$ provided n_1 is large enough. Integrating the above inequality over $[0, r]$, we get

$$\frac{d}{dr}\Gamma(U_n) \geq -r^{1-N}\int_0^r s^{N-1}p(s)\,ds.$$

Integrating this new inequality over $[r, R_n]$ we obtain

$$-\Gamma(U_n(r)) \geq -\int_r^{R_n} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt, \qquad \forall n \geq n_1,$$

since $U_n(r) \to \infty$ as $r \nearrow R_n$ implies $\Gamma(U_n(r)) \to 0$ as $r \nearrow R_n$. Therefore,

$$\Gamma(U_n(r)) \leq \int_r^{R_n} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt, \qquad \forall n \geq n_1.$$

Letting $n \to \infty$ and using (32) we find

$$\Gamma(U(r)) \leq \int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt,$$

or, equivalently

$$U(r) \geq \Gamma^{-1}\left(\int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt\right).$$

Passing to the limit as $r \nearrow R$ and using the fact that $\lim_{s\searrow 0}\Gamma^{-1}(s) = \infty$ we deduce

$$\lim_{r\nearrow R} U(r) \geq \lim_{r\nearrow R}\Gamma^{-1}\left(\int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt\right) = \infty.$$

But (U, V) is an entire solution so that we conclude $R = \infty$ and $\lim_{r\to\infty} U(r) = \infty$. Since (3) holds and $V'(r) \geq 0$ we find

$$\begin{aligned}U(r) &\leq a + g(V(r))\int_0^\infty t^{1-N}\int_0^t s^{N-1}p(s)\,ds\,dt \\ &\leq a + g(V(r))\frac{1}{N-2}\int_0^\infty tp(t)\,dt, \qquad \forall r \geq 0.\end{aligned}$$

We deduce $\lim_{r\to\infty} V(r) = \infty$, otherwise we obtain that $\lim_{r\to\infty} U(r)$ is finite, a contradiction. Consequently, (U, V) is an entire large solution of (1). This concludes our proof. ■

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Uniqueness of the blow-up boundary solution of logistic equations with absorbtion *

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Abstract. Let Ω be a smooth bounded domain in $\mathbf{R}^N$. Assume $f \in C^1[0,\infty)$ is a non-negative function such that $f(u)/u$ is increasing on $(0,\infty)$. Let a be a real number and let $b \geq 0$, $b \not\equiv 0$ be a continuous function such that $b \equiv 0$ on $\partial\Omega$. We study the logistic equation $\Delta u + au = b(x)f(u)$ in Ω. The special feature of this work is the uniqueness of positive solutions blowing-up on $\partial\Omega$, in a general setting that arises in probability theory.

Unicité de la solution explosant au bord pour équations logistiques avec absorption

Résumé. *Soit Ω un domaine borné et régulier de $\mathbf{R}^N$. On suppose que $f \in C^1[0,\infty)$ est une fonction non-negative telle que $f(u)/u$ soit strictement croissante sur $(0,+\infty)$. Soit a un réel et $b \geq 0$, $b \not\equiv 0$, une fonction continue sur $\overline{\Omega}$ telle que $b \equiv 0$ sur $\partial\Omega$. On étudie l'équation logistique $\Delta u + au = b(x)f(u)$ sur Ω. Le but de cette Note est de montrer l'unicité de la solution explosant au bord de Ω dans un contexte général, qui apparaît en théorie des probabilités.*

Version française abrégée. Soit $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ un domaine borné et régulier, a un paramètre réel et $b \in C^{0,\mu}(\overline{\Omega})$, $\mu \in (0,1)$, $b \geq 0$, $b \not\equiv 0$ dans Ω. On considère l'équation logistique

$$\Delta u + au = b(x)f(u) \qquad \text{dans } \Omega, \tag{1}$$

où $f \in C^1[0,\infty)$ satisfait

(A_1) $\quad f \geq 0$ et $f(u)/u$ est strictement croissante sur $(0,+\infty)$.

Soit

$$\Omega_0 := \operatorname{int}\{x \in \Omega : \ b(x) = 0\}$$

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et on suppose que $\partial\Omega_0$ est régulier (éventuellement vide), $\overline{\Omega}_0 \subset \Omega$ et $b > 0$ sur $\Omega \setminus \overline{\Omega}_0$. On désigne par $\lambda_{\infty,1}$ la première valeur propre (avec conditions de Dirichlet) de l'opérateur $(-\Delta)$ dans Ω_0, avec la convention $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$.

On dit que u est une solution *large (explosive)* de (1) si $u \geq 0$ dans Ω et $u(x) \to \infty$ si $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0$.

Soit $D > 0$ et $R : [D, \infty) \to (0, +\infty)$ une fonction mesurable. On dit que R a une variation régulière d'indice $\rho \in \mathbf{R}$ (notation: $R \in \mathbf{R}_\rho$) si $\lim_{u\to\infty} R(\xi u)/R(u) = \xi^\rho$, pour chaque $\xi > 0$ (voir [11]).

Soit $\mathcal{K}$ l'ensemble des fonctions $k : (0, \nu) \to (0, +\infty)$ (pour un certain ν), de classe C^1, croissantes, telles que $\lim_{t\to 0^+} \left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)^{(i)} := \ell_i$, pour $i = \overline{0,1}$.

On démontre le résultat suivant.

Théorème 1. - *Supposons que la fonction f satisfait la condition (A_1) et que f' est une fonction à variation régulière d'indice $\rho \neq 0$. De plus, on suppose que le potentiel b vérifie*

(B) $\quad b(x) = c\,k^2(d(x)) + o(k^2(d(x)))$ *si* $d(x) \to 0$, *avec* $c > 0$ *et* $k \in \mathcal{K}$.

Alors, pour chaque $a \in (-\infty, \lambda_{\infty,1})$, l'équation (1) admet une unique solution explosive u_a. On a, de plus,

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0,$$

où $\xi_0 = \left(\frac{2 + \ell_1\rho}{c(2+\rho)}\right)^{1/\rho}$ *et la fonction h est définie par*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0, \nu).$$

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ be a smooth bounded domain. Consider the semilinear elliptic equation

$$\Delta u + au = b(x)f(u) \qquad \text{in } \Omega, \tag{1}$$

where a is a real parameter and $b \in C^{0,\mu}(\overline{\Omega})$, for some $\mu \in (0,1)$, such that $b \geq 0$, $b \not\equiv 0$ in Ω.

Suppose that $f \in C^1[0,\infty)$ satisfies

(A_1) $\quad f \geq 0$ and $f(u)/u$ is increasing on $(0, \infty)$.

In the study of positive solutions for (1), subject to the homogeneous Dirichlet boundary condition, an important role is played by the zero set (see [1])

$$\Omega_0 := \operatorname{int}\{x \in \Omega : \ b(x) = 0\}.$$

We shall assume throughout that Ω_0 is smooth (possibly empty), $\overline{\Omega}_0 \subset \Omega$, and $b > 0$ in $\Omega \setminus \overline{\Omega}_0$.

By a *large (explosive)* solution of (1) we mean a solution u of (1) such that $u \geq 0$ in Ω and $u(x) \to \infty$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0$. In [3, 4] we study the existence of large solutions for (1) and also deduce several existence and unicity results for a related problem. Note that any large solution of (1) is *positive* and it can exists only if the Keller-Osserman condition holds (see [4])

$$(A_2) \quad \int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty\,, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$$

Let H_∞ define the Dirichlet Laplacian on the set $\Omega_0 \subset \Omega$ as the unique self-adjoint operator associated to the quadratic form $\psi(u) = \int_\Omega |\nabla u|^2\,dx$ with form domain

$$H_D^1(\Omega_0) = \{u \in H_0^1(\Omega) : \; u(x) = 0 \quad \text{for a.e. } x \in \Omega \setminus \Omega_0\}.$$

If $\partial\Omega_0$ satisfies an exterior cone condition, then $H_D^1(\Omega_0)$ coincides with $H_0^1(\Omega_0)$ and H_∞ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of H_∞ in Ω_0. We understand $\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$.

The main result in [3] asserts that equation (1) has a large solution iff $a \in (-\infty, \lambda_{\infty,1})$.

The special feature of this paper is the uniqueness of large solutions of (1) in a general framework for f and b, under the restriction $b \equiv 0$ on $\partial\Omega$, inherited from the logistic equation (see [6]).

We start with

DEFINITION 1 ([11]). - A positive measurable function R defined on $[D, \infty)$, for some $D > 0$, is called *regularly varying (at infinity) with index* $q \in \mathbf{R}$, written $R \in \mathbf{R}_q$, if for all $\xi > 0$

$$\lim_{u\to\infty} R(\xi u)/R(u) = \xi^q.$$

When the index of regular variation q is zero, we say that the function is *slowly varying*.

REMARK 1. - *Any function $R \in \mathbf{R}_q$ can be written in terms of a slowly varying function.* Indeed, set $R(u) = u^q L(u)$. From Definition 1 we easily derive that L varies slowly.

The canonical q-varying function is u^q. The functions $\ln(1+u)$, $\ln\ln(e+u)$, $\exp\{(\ln u)^\alpha\}$, $\alpha \in (0,1)$ vary slowly, as well as any measurable function on $[D, \infty)$ with positive limit at infinity.

In what follows L denotes an arbitrary slowly varying function and $D > 0$ a positive number. For details on Properties 1-4 stated below, we refer to Seneta [11] (pp. 7, 18, 53 and 78).

PROPERTY 1. - For any $m > 0$, $u^m L(u) \to \infty$, $\;u^{-m} L(u) \to 0$ as $u \to \infty$.

PROPERTY 2. - Any positive C^1-function on $[D, \infty)$ satisfying $uL_1'(u)/L_1(u) \to 0$ as $u \to \infty$ is slowly varying. Moreover, if the above limit is $q \in \mathbf{R}$, then $L_1 \in \mathbf{R}_q$.

PROPERTY 3. - Assume $R : [D, \infty) \to (0, \infty)$ is measurable and Lebesgue integrable on each finite subinterval of $[D, \infty)$. Then R varies regularly iff there exists $j \in \mathbf{R}$ such that

$$\lim_{u\to\infty} \frac{u^{j+1} R(u)}{\int_D^u x^j R(x)\,dx} \tag{2}$$

exists and is a positive number, say $a_j + 1$. In this case, $R \in \mathbf{R}_q$ with $q = a_j - j$.

PROPERTY 4 (Karamata Theorem, 1933). - If $R \in \mathbf{R}_q$ is Lebesgue integrable on each finite subinterval of $[D, \infty)$, then the limit defined by (2) is $q + j + 1$, for every $j > -q - 1$.

LEMMA 1. - *Assume (A_1) holds. Then we have the equivalence*

$$a)\ f' \in \mathbf{R}_\rho \iff b)\ \lim_{u\to\infty} uf'(u)/f(u) := \vartheta < \infty \iff c)\ \lim_{u\to\infty} (F/f)'(u) := \gamma > 0.$$

REMARK 2. - *Let a) of Lemma 1 be fulfilled. The following assertions hold*
(*i*) ρ *is* **non-negative**. Indeed, if $\rho < 0$ then Property 1 and Remark 1 would contradict (A_1);
(*ii*) $\gamma = 1/(\rho+2) = 1/(\vartheta+1)$ (see the proof of Lemma 1);
(*iii*) *If* $\rho \neq 0$, *then* (A_2) *holds* (use $\lim_{u\to\infty} f(u)/u^p = \infty$, $\forall p \in (1, 1+\rho)$). *The converse implication is not necessarily true (take* $f(u) = u\ln^4(u+1)$*). However, there are cases when* $\rho = 0$ *and* (A_2) *fails so that (1) has* **no** *large solutions. This is illustrated by* $f(u) = u$ *or* $f(u) = u\ln(u+1)$.

Inspired by the definition of γ, we denote by $\mathcal{K}$ the set of all positive, increasing C^1-functions k defined on $(0,\nu)$, for some $\nu > 0$, which satisfy $\lim_{t\to 0^+}\left(\dfrac{\int_0^t k(s)\,ds}{k(t)}\right)^{(i)} := \ell_i,\ i = \overline{0,1}$.

It is easy to see that $\ell_0 = 0$ and $\ell_1 \in [0,1]$, for every $k \in \mathcal{K}$. Our next result gives examples of functions $k \in \mathcal{K}$ with $\lim_{t\to 0^+} k(t) = 0$, for *every* $\ell_1 \in [0,1]$.

LEMMA 2. - *Let* $S \in C^1[D,\infty)$ *be such that* $S' \in \mathbf{R}_q$ *with* $q > -1$. *Hence the following hold:*
a) *If* $k(t) = \exp\{-S(1/t)\}$ $\forall t \le 1/D$, *then* $k \in \mathcal{K}$ *with* $\ell_1 = 0$.
b) *If* $k(t) = 1/S(1/t)$ $\forall t \le 1/D$, *then* $k \in \mathcal{K}$ *with* $\ell_1 = 1/(q+2) \in (0,1)$.
c) *If* $k(t) = 1/\ln S(1/t)$ $\forall t \le 1/D$, *then* $k \in \mathcal{K}$ *with* $\ell_1 = 1$.

REMARK 3. - *If* $S \in C^1[D,\infty)$, *then* $S' \in \mathbf{R}_q$ *with* $q > -1$ *iff for some* $m > 0$, $C > 0$ *and* $B > D$ *we have* $S(u) = Cu^m \exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\}$, $\forall u \ge B$, *where* $y \in C[B,\infty)$ *satisfies* $\lim_{u\to\infty} y(u) = 0$. *In this case,* $S' \in \mathbf{R}_q$ *with* $q = m-1$. This is a consequence of Properties 3 and 4.

Our main result is

THEOREM 1. - *Let* (A_1) *hold and* $f' \in \mathbf{R}_\rho$ *with* $\rho > 0$. *Assume* $b \equiv 0$ *on* $\partial\Omega$ *satisfies*

(B) $\quad b(x) = c\,k^2(d(x)) + o(k^2(d(x)))$ *as* $d(x) \to 0$, *for some constant* $c > 0$ *and* $k \in \mathcal{K}$.

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, *Eq. (1) admits a unique large solution* u_a. *Moreover,*

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \tag{3}$$

where $\xi_0 = \left(\dfrac{2+\ell_1\rho}{c(2+\rho)}\right)^{1/\rho}$ *and* h *is defined by*

$$\int_{h(t)}^\infty \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0,\nu). \tag{4}$$

By Remark 3, the assumption $f' \in \mathbf{R}_\rho$ with $\rho > 0$ holds iff there exist $p > 1$ and $B > 0$ such that $f(u) = Cu^p\exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\}$, for all $u \ge B$ (y as before and $p = \rho+1$). If B is large enough ($y > -\rho$ on $[B,\infty)$), then $f(u)/u$ is increasing on $[B,\infty)$. Thus, to get the whole range of functions f for which our Theorem 1 applies we have only to "paste" a suitable smooth function on $[0,B]$ in accordance with (A_1). A simple way to do this is to define $f(u) = u^p\exp\{\int_0^u \frac{z(t)}{t}\,dt\}$, for all $u \ge 0$, where $z \in C[0,\infty)$ is non-negative such that $\lim_{t\to 0^+} z(t)/t \in [0,\infty)$ and $\lim_{u\to\infty} z(u) = 0$. Clearly, $f(u) = u^p$, $f(u) = u^p\ln(u+1)$, and $f(u) = u^p\arctan u$ ($p > 1$) fall into this category.

Lemma 2 provides a practical method to find functions k which can be considered in the statement of Theorem 1. Here are some examples: $k(t) = \exp\{-1/t^\alpha\}$, $k(t) = \exp\{-\ln(1+\frac{1}{t})/t^\alpha\}$, $k(t) = \exp\{-\left[\arctan\left(\frac{1}{t}\right)\right]/t^\alpha\}$, $k(t) = -1/\ln t$, $k(t) = t^\alpha/\ln(1+\frac{1}{t})$, $k(t) = t^\alpha$, for some $\alpha > 0$.

As we shall see, the uniqueness lies upon the crucial observation (3), which shows that all explosive solutions have the same boundary behaviour. Note that the only case of Theorem 1 studied so far is $f(u) = u^p$ $(p > 1)$ and $k(t) = t^\alpha$ $(\alpha > 0)$ (see [6]). For related results on the uniqueness of explosive solutions (mainly in the cases $b \equiv 1$ and $a = 0$) we refer to [2, 8, 9, 12].

Proof of Lemma 1. - From Property 4 and Remark 2 (i) we deduce $a) \Longrightarrow b)$ and $\vartheta = \rho + 1$. Conversely, $b) \Longrightarrow a)$ follows by Property 3 since $\vartheta \geq 1$ cf. (A_1).

$b) \Longrightarrow c)$. Indeed, $\lim_{u\to\infty} \frac{uf(u)}{F(u)} = 1 + \vartheta$, which yields $\frac{\vartheta}{1+\vartheta} = \lim_{u\to\infty} \left[1 - \left(\frac{F}{f}\right)'(u)\right] = 1 - \gamma$.

$c) \Longrightarrow b)$. Choose $s_1 > 0$ such that $\left(\frac{F}{f}\right)'(u) \geq \frac{\gamma}{2}$, $\forall u \geq s_1$. So, $\left(\frac{F}{f}\right)(u) \geq \frac{(u-s_1)\gamma}{2} + \left(\frac{F}{f}\right)(s_1)$, $\forall u \geq s_1$. Passing to the limit $u \to \infty$, we find $\lim_{u\to\infty} \frac{F(u)}{f(u)} = \infty$. Thus, $\lim_{u\to\infty} \frac{uf(u)}{F(u)} = \frac{1}{\gamma}$. Since $1 - \gamma := \lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)}$, we obtain $\lim_{u\to\infty} \frac{uf'(u)}{f(u)} = \frac{1-\gamma}{\gamma}$. ■

Proof of Lemma 2. - Since $\lim_{u\to\infty} uS'(u) = \infty$ (cf. Property 1), from Karamata Theorem we deduce $\lim_{u\to\infty} \frac{uS'(u)}{S(u)} = q + 1 > 0$. Therefore, in any of the cases $a)$, $b)$, $c)$, $\lim_{t\to 0^+} k(t) = 0$ and k is an increasing C^1-function on $(0, \nu)$, for $\nu > 0$ sufficiently small.

$a)$ It is clear that $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)\ln k(t)} = \lim_{t\to 0^+} \frac{-S'(1/t)}{tS(1/t)} = -(q+1)$. By l'Hospital's rule, $\ell_0 = \lim_{t\to 0^+} \frac{k(t)}{k'(t)} = 0$ and $\lim_{t\to 0^+} \frac{\left(\int_0^t k(s)\,ds\right)\ln k(t)}{tk(t)} = -\frac{1}{q+1}$. So, $1 - \ell_1 := \lim_{t\to 0^+} \frac{\left(\int_0^t k(s)\,ds\right)k'(t)}{k^2(t)} = 1$.

$b)$ We see that $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \lim_{t\to 0^+} \frac{S'(1/t)}{tS(1/t)} = q + 1$. By l'Hospital's rule, $\ell_0 = 0$ and $\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{tk(t)} = \frac{1}{q+2}$. So, $\ell_1 = 1 - \lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{tk(t)} \frac{tk'(t)}{k(t)} = \frac{1}{q+2}$.

$c)$ We have $\lim_{t\to 0^+} \frac{tk'(t)}{k^2(t)} = \lim_{t\to 0^+} \frac{S'(1/t)}{tS(1/t)} = q+1$. By l'Hospital's rule, $\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{tk(t)} = 1$. Thus, $\ell_0 = 0$ and $\ell_1 = 1 - \lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{t} \frac{tk'(t)}{k^2(t)} = 1$. ■

Proof of Theorem 1. - Fix $a \in (-\infty, \lambda_{\infty,1})$. By [3, Theorem 1], (1) has at least a large solution.

If we prove that (3) holds for an *arbitrary* large solution u_a of (1), then the uniqueness is a consequence of [3, Lemma 3]. Indeed, if u_1 and u_2 are two arbitrary large solutions of (1), then (3) yields $\lim_{d(x)\to 0^+} \frac{u_1(x)}{u_2(x)} = 1$. Hence, for any $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$(1-\varepsilon)u_2(x) \leq u_1(x) \leq (1+\varepsilon)u_2(x), \quad \forall x \in \Omega \text{ with } 0 < d(x) \leq \delta. \tag{5}$$

Choosing eventually a smaller $\delta > 0$, we can assume that $\overline{\Omega}_0 \subset C_\delta$, where $C_\delta := \{x \in \Omega : \, d(x) > \delta\}$.

It is clear that u_1 is a positive solution of the boundary value problem

$$\Delta\phi + a\phi = b(x)f(\phi) \quad \text{in } C_\delta, \qquad \phi = u_1 \quad \text{on } \partial C_\delta. \tag{6}$$

By (A_1) and (5), we see that $\phi^- = (1-\varepsilon)u_2$ (resp., $\phi^+ = (1+\varepsilon)u_2$) is a positive sub-solution (resp., super-solution) of (6). By the sub and super-solutions method, (6) has a positive solution ϕ_1 satisfying $\phi^- \leq \phi_1 \leq \phi^+$ in C_δ. Since $b > 0$ on $\overline{C_\delta} \setminus \overline{\Omega}_0$, by [3, Lemma 3] we derive that (6) has a *unique* positive solution, i.e., $u_1 \equiv \phi_1$ in C_δ. This yields $(1-\varepsilon)u_2(x) \leq u_1(x) \leq (1+\varepsilon)u_2(x)$ in C_δ, so that (5) holds in Ω. Passing to the limit $\varepsilon \to 0^+$, we conclude that $u_1 \equiv u_2$.

In order to prove (3) we state some useful properties about h:

(h_1) $h \in C^2(0,\nu)$, $\lim_{t\to 0^+} h(t) = \infty$ (straightforward from (4)).

(h_2) $\lim_{t\to 0^+} \dfrac{h''(t)}{k^2(t)f(h(t)\xi)} = \dfrac{1}{\xi^{\rho+1}} \dfrac{2+\rho\ell_1}{2+\rho}$, $\forall \xi > 0$ (so, $h'' > 0$ on $(0, 2\delta)$, for $\delta > 0$ small enough).

(h_3) $\lim_{t\to 0^+} h(t)/h''(t) = \lim_{t\to 0^+} h'(t)/h''(t) = 0$.

We check (h_2) for $\xi = 1$ only, since $f \in \mathbf{R}_{\rho+1}$. Clearly, $h'(t) = -k(t)\sqrt{2F(h(t))}$ and

$$h''(t) = k^2(t) f(h(t)) \left(1 - 2\frac{k'(t)\left(\int_0^t k(s)\,ds\right)}{k^2(t)} \frac{\sqrt{F(h(t))}}{f(h(t))\int_{h(t)}^{\infty}[F(s)]^{-1/2}ds}\right) \quad \forall t \in (0,\nu). \tag{7}$$

We see that $\lim_{u\to\infty}\sqrt{F(u)}/f(u) = 0$. Thus, from l'Hospital's rule and Lemma 1 we infer that

$$\lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u)\int_u^{\infty}[F(s)]^{-1/2}ds} = \frac{1}{2} - \gamma = \frac{\rho}{2(\rho+2)}. \tag{8}$$

Using (7) and (8) we derive (h_2) and also

$$\lim_{t\to 0^+} \frac{h'(t)}{h''(t)} = \frac{-2(2+\rho)}{2+\ell_1\rho} \lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{k(t)} \lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u)\int_u^{\infty}[F(s)]^{-1/2}ds} = \frac{-\rho\ell_0}{2+\ell_1\rho} = 0. \tag{9}$$

From (h_1) and (h_2), $\lim_{t\to 0^+} h'(t) = -\infty$. So, l'Hospital's rule and (9) yield $\lim_{t\to 0^+}\frac{h(t)}{h'(t)} = 0$. This and (9) lead to $\lim_{t\to 0^+}\frac{h(t)}{h''(t)} = 0$ which proves (h_3).

Proof of (3). Fix $\varepsilon \in (0, c/2)$. Since $b \equiv 0$ on $\partial\Omega$ and (B) holds, we take $\delta > 0$ so that
(i) $d(x)$ is a C^2-function on the set $\{x \in \mathbf{R}^N :\ d(x) < 2\delta\}$;
(ii) k^2 is increasing on $(0, 2\delta)$;
(iii) $(c-\varepsilon)k^2(d(x)) < b(x) < (c+\varepsilon)k^2(d(x))$, $\forall x \in \Omega$ with $0 < d(x) < 2\delta$;
(iv) $h''(t) > 0$ $\forall t \in (0, 2\delta)$ (from (h_2)).

Let $\sigma \in (0,\delta)$ be arbitrary. We define $\xi^{\pm} = \left[\frac{2+\ell_1\rho}{(c\mp 2\varepsilon)(2+\rho)}\right]^{1/\rho}$ and $v_\sigma^-(x) = h(d(x)+\sigma)\xi^-$, for all x with $d(x)+\sigma < 2\delta$ resp., $v_\sigma^+(x) = h(d(x)-\sigma)\xi^+$, for all x with $\sigma < d(x) < 2\delta$.

Using (i)-(iv), when $\sigma < d(x) < 2\delta$ we obtain (since $|\nabla d(x)| \equiv 1$)

$$\begin{aligned}\Delta v_\sigma^+ + av_\sigma^+ - b(x)f(v_\sigma^+) \le \xi^+ h''(d(x)-\sigma) \Bigg(& \frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\Delta d(x) + a\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} + 1 \\ & -(c-\varepsilon)\frac{k^2(d(x)-\sigma)f(h(d(x)-\sigma)\xi^+)}{h''(d(x)-\sigma)\xi^+}\Bigg).\end{aligned}$$

Similarly, when $d(x)+\sigma < 2\delta$ we find

$$\begin{aligned}\Delta v_\sigma^- + av_\sigma^- - b(x)f(v_\sigma^-) \ge \xi^- h''(d(x)+\sigma) \Bigg(& \frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\Delta d(x) + a\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} + 1 \\ & -(c+\varepsilon)\frac{k^2(d(x)+\sigma)f(h(d(x)+\sigma)\xi^-)}{h''(d(x)+\sigma)\xi^-}\Bigg).\end{aligned}$$

Using (h_2) and (h_3) we see that, by diminishing δ, we can assume

$$\Delta v_\sigma^+(x) + av_\sigma^+(x) - b(x)f(v_\sigma^+(x)) \le 0 \quad \forall x \text{ with } \sigma < d(x) < 2\delta;$$

$$\Delta v_\sigma^-(x) + av_\sigma^-(x) - b(x)f(v_\sigma^-(x)) \ge 0 \quad \forall x \text{ with } d(x)+\sigma < 2\delta.$$

Let Ω_1 and Ω_2 be smooth bounded domains such that $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2$ and the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $\Omega_1\setminus\overline{\Omega}$ is greater than a. Let $p \in C^{0,\mu}(\overline{\Omega}_2)$ satisfy $0 < p(x) \le b(x)$ for $x \in \Omega \setminus C_{2\delta}$, $p = 0$ on $\overline{\Omega}_1 \setminus \Omega$ and $p > 0$ on $\Omega_2 \setminus \overline{\Omega}_1$. Denote by w a positive large solution of

$$\Delta w + aw = p(x)f(w) \quad \text{in } \Omega_2 \setminus \overline{C}_{2\delta}.$$

The existence of w is ensured by Theorem 1 in [3].

Suppose that u_a is an arbitrary large solution of (1) and let $v := u_a + w$. Then v satisfies

$$\Delta v + av - b(x)f(v) \le 0 \quad \text{in } \Omega \setminus \overline{C}_{2\delta}.$$

Since $v_{|\partial\Omega} = \infty > v^-_{\sigma|\partial\Omega}$ and $v_{|\partial C_{2\delta}} = \infty > v^-_{\sigma|\partial C_{2\delta}}$, Lemma 1 in [3] implies

$$u_a + w \ge v_\sigma^- \quad \text{on } \Omega \setminus \overline{C}_{2\delta}. \tag{10}$$

Similarly,

$$v_\sigma^+ + w \ge u_a \quad \text{on } C_\sigma \setminus \overline{C}_{2\delta}. \tag{11}$$

Letting $\sigma \to 0$ in (10) and (11), we deduce $h(d(x))\xi^+ + 2w \ge u_a + w \ge h(d(x))\xi^-$, for all $x \in \Omega \setminus \overline{C}_{2\delta}$. Since w is uniformly bounded on $\partial\Omega$, we have $\xi^- \le \liminf_{d(x)\to 0} \dfrac{u_a(x)}{h(d(x))} \le \limsup_{d(x)\to 0} \dfrac{u_a(x)}{h(d(x))} \le \xi^+$. Letting $\varepsilon \to 0^+$ we obtain (3). This concludes the proof of Theorem 1. ■

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Asymptotics for the blow-up boundary solution of the logistic equation with absorption*

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Abstract. Let Ω be a smooth bounded domain in $\mathbf{R}^N$. Assume that $f \geq 0$ is a C^1-function on $[0,\infty)$ such that $f(u)/u$ is increasing on $(0,+\infty)$. Let a be a real number and let $b \geq 0$, $b \not\equiv 0$ be a continuous function such that $b \equiv 0$ on $\partial\Omega$. The purpose of this Note is to establish the asymptotic behaviour of the unique positive solution of the logistic problem $\Delta u + au = b(x)f(u)$ in Ω, subject to the singular boundary condition $u(x) \to +\infty$ as dist $(x, \partial\Omega) \to 0$. Our analysis is based on the Karamata regular variation theory.

Comportement asymptotique de la solution explosant au bord de l'équation logistique avec absorption

Résumé. *Soit Ω un domaine borné et régulier de $\mathbf{R}^N$. On suppose que $0 \leq f \in C^1[0,\infty)$ est telle que $f(u)/u$ soit strictement croissante sur $(0,+\infty)$. Soit a un réel et $b \geq 0$, $b \not\equiv 0$, une fonction continue sur $\overline{\Omega}$ telle que $b \equiv 0$ sur $\partial\Omega$. Dans cette Note on établit le comportement asymptotique de l'unique solution positive du problème logistique $\Delta u + au = b(x)f(u)$ sur Ω avec la donnée au bord singulière $u(x) \to +\infty$ si* dist $(x, \partial\Omega) \to 0$. *Notre analyse porte sur la théorie de la variation régulière de Karamata.*

Version française abrégée. Soit $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ un domaine borné et régulier, a un paramètre réel et $0 \not\equiv b \in C^{0,\mu}(\overline{\Omega})$, $b \geq 0$ dans Ω. On considère le problème logistique avec explosion au bord

$$\Delta u + au = b(x)f(u) \ \text{ dans } \Omega, \qquad u(x) \to +\infty \ \text{ si } d(x) := \text{dist}\,(x, \partial\Omega) \to 0, \tag{1}$$

où $0 \leq f \in C^1[0,\infty)$ satisfait la condition de Keller–Osserman (voir [6, 7]) et telle que $f(u)/u$ soit strictement croissante sur $(0,+\infty)$. Soit $\Omega_0 := \text{int}\,\{x \in \Omega : \ b(x) = 0\}$. On suppose que $\partial\Omega_0$ est régulier (éventuellement vide), $\overline{\Omega}_0 \subset \Omega$ et $b > 0$ sur $\Omega \setminus \overline{\Omega}_0$. On désigne par $\lambda_{\infty,1}$ la première valeur propre de l'opérateur $(-\Delta)$ dans $H_0^1(\Omega_0)$, avec la convention $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$. Dans [2] on

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montre que le problème (1) admet une solution positive u_a si et seulement si $a < \lambda_{\infty,1}$. L'unicité de la solution u_a est établie dans [1]. Soit $\mathcal{K}$ l'ensemble des fonctions $k : (0,\nu) \to (0,\infty)$ (pour un certain ν), de classe C^1, croissantes, telles que $\lim_{t\searrow 0}(\int_0^t k(s)\,ds/k(t))^{(i)} := \ell_i$, pour $i = \overline{0,1}$.

Soit RV_q ($q \in \mathbf{R}$) l'ensemble des fonctions positives et mesurables $Z : [A,\infty) \to \mathbf{R}$ (avec $A > 0$) telles que $\lim_{u\to\infty} Z(\xi u)/Z(u) = \xi^q$, $\forall \xi > 0$. On désigne par NRV_q la classe des fonctions f définies par $f(u) = Cu^q \exp\left\{\int_B^u \phi(t)/t\,dt\right\}$, $\forall u \geq B > 0$, où $C > 0$ et $\phi \in C[B,\infty)$ satisfait $\lim_{t\to\infty}\phi(t) = 0$. Supposons que $0 \leq f \in C^1[0,\infty) \cap NRV_{\rho+1}$ ($\rho > 0$) est telle que $f(u)/u$ soit strictement croissante sur $(0,\infty)$ et que $b \equiv 0$ sur $\partial\Omega$ vérifie $b(x) = k^2(d)(1+o(1))$ si $d(x) \to 0$, avec $k \in \mathcal{K}$. Alors, pour chaque $a < \lambda_{\infty,1}$, le problème (1) admet une unique solution positive u_a (voir [1]). Le but de cette Note est d'établir la vitesse d'explosion au bord de la solution u_a.

Pour chaque $\zeta > 0$, soit

$$\mathcal{R}_{0,\zeta} = \left\{ \begin{array}{l} k: \quad k(u^{-1}) = d_0 u\,[\Lambda(u)]^{-1} \exp\left[-\int_{d_1}^u (s\Lambda(s))^{-1}\,ds\right] \ (u \geq d_1),\ 0 < \Lambda \in C^1[d_1,\infty), \\ \qquad \lim_{u\to\infty}\Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0,\ \lim_{u\to\infty} u^{\zeta+1}\Lambda'(u) = \ell_\star \in \mathbf{R},\ d_0,\ d_1 > 0 \end{array} \right\}.$$

On a $\mathcal{R}_{0,\zeta} \subset \mathcal{K}$. De plus, si $k \in \mathcal{R}_{0,\zeta}$ alors $\ell_1 = 0$ et $\lim_{t\to 0} k(t) = 0$.

On définit les classes $\mathcal{F}_{\rho\eta} = \{f \in NRV_{\rho+1}(\rho > 0) :\ \phi \in RV_\eta \text{ ou } -\phi \in RV_\eta\}$, si $\eta \in (-\rho-2, 0]$ et $\mathcal{F}_{\rho 0,\tau} = \{f \in \mathcal{F}_{\rho 0} :\ \lim_{u\to\infty}(\ln u)^\tau \phi(u) = \ell^\star \in \mathbf{R}\}$, pour $\tau \in (0,\infty)$.

On démontre le résultat suivant.

THÉORÈME 1. - *On suppose que $b(x) = k^2(d)(1+\tilde{c}d^\theta + o(d^\theta))$ si $d(x) \to 0$ (avec $\theta > 0$, $\tilde{c} \in \mathbf{R}$), où $k \in \mathcal{R}_{0,\zeta}$. Soit $0 \leq f \in C^1[0,\infty)$ telle que $f(u)/u$ soit strictement croissante sur $(0,\infty)$. De plus, on suppose que f satisfait l'un des cas suivants de croissance à l'infini:*

(i) *$f(u) = Cu^{\rho+1}$ dans un voisinage de l'infini;*

(ii) *$f \in \mathcal{F}_{\rho\eta}$ avec $\eta \neq 0$;*

(iii) *$f \in \mathcal{F}_{\rho 0,\tau_1}$ avec $\tau_1 = \varpi/\zeta$, où $\varpi = \min\{\theta,\zeta\}$.*

Alors, pour chaque $a \in (-\infty, \lambda_{\infty,1})$, l'unique solution positive u_a du problème (1) satisfait

$$u_a(x) = \xi_0 h(d)(1 + \chi d^\varpi + o(d^\varpi)) \quad \text{si} \quad d(x) \to 0,$$

où $\xi_0 = [2(2+\rho)^{-1}]^{1/\rho}$ et h est définie par $\int_{h(t)}^\infty [2F(s)]^{-1/2} ds = \int_0^t k(s) ds$, pour $t > 0$ suffisement petit. L'expression de χ est donnée par

$$\chi = \left\{ \begin{array}{l} -(1+\zeta)\ell_\star(2\zeta)^{-1}\text{Heaviside}(\theta-\zeta) - \tilde{c}\rho^{-1}\text{Heaviside}(\zeta-\theta) = \chi_1 \ \textit{dans les cas (i) et (ii)} \\ \chi_1 - \ell^\star\rho^{-1}(-\rho\ell_\star/2)^{\tau_1}\,(1/(\rho+2) + \ln\xi_0) \ \textit{pour le cas (iii)}. \end{array} \right.$$

Notons que le seul cas lié à ce résultat et correspondant à la situation particulière $\Omega_0 = \emptyset$, $f(u) = u^{\rho+1}$, $k(t) = ct^\alpha \in \mathcal{K}$ (avec $c, \alpha > 0$), $\theta = 1$, a été étudié dans [4]. Dans ce travail, les deux premiers termes du développement asymptotique de u_a autour de $\partial\Omega$ tiennent compte de $d(x)$ ainsi que de la courbure moyenne H de $\partial\Omega$. Dans notre résultat on enlève la restriction $b > 0$ dans Ω et on garde la condition $b \equiv 0$ sur $\partial\Omega$, comme restriction naturelle héritée du problème logistique (voir [4]). De plus, on raffine la vitesse d'explosion de u_a pour une large classe de potentiels b, avec $\theta > 0$ quelconque et k appartenant à un ensemble très riche de fonctions.

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ be a smooth bounded domain. Consider the blow-up logistic problem

$$\Delta u + au = b(x)f(u) \text{ in } \Omega, \qquad u(x) \to +\infty \text{ as } d(x) := \mathrm{dist}\,(x, \partial\Omega) \to 0, \tag{1}$$

where $f \in C^1[0,\infty)$, a is a real parameter and $0 \not\equiv b \in C^{0,\mu}(\overline{\Omega})$ (for some $\mu \in (0,1)$) satisfies $b \geq 0$ in Ω. Suppose that the absorption term f fulfills both

(A) $\quad f \geq 0$ and $f(u)/u$ is increasing on $(0,\infty)$

and the Keller–Osserman condition (see [6, 7]) $\int_1^\infty [F(t)]^{-1/2}\,dt < \infty$, where $F(t) = \int_0^t f(s)\,ds$.

Assume throughout that $\Omega_0 \subset\subset \Omega$ satisfies the exterior cone condition (possibly, $\Omega_0 = \emptyset$) and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$, where $\Omega_0 := \mathrm{int}\,\{x \in \Omega : \ b(x) = 0\}$. Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(\Omega_0)$. Set $\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$. Under the above assumptions, we have proved in [2] that (1) has a positive solution u_a if and only if $a < \lambda_{\infty,1}$. Moreover, the uniqueness of u_a is studied in [1]. Denote by $\mathcal{K}$ the set of all positive increasing C^1-functions k defined on $(0,\nu)$, for some $\nu > 0$, which satisfy $\lim_{t\searrow 0}(\int_0^t k(s)\,ds/k(t))^{(i)} := \ell_i$, $i \in \overline{0,1}$. We have $\ell_0 = 0$ and $\ell_1 \in [0,1]$.

Let us now recall some basic definitions related to the Karamata regular variation theory (see [5, 8]). Let RV_q $(q \in \mathbf{R})$ be the set of all positive measurable functions $Z : [A,\infty) \to \mathbf{R}$ (for some $A > 0$) satisfying $\lim_{u\to\infty} Z(\xi u)/Z(u) = \xi^q$, $\forall \xi > 0$. Define by NRV_q the class of functions f in the form $f(u) = Cu^q \exp\left\{\int_B^u \phi(t)/t\,dt\right\}$, $\forall u \geq B > 0$, where $C > 0$ is a constant and $\phi \in C[B,\infty)$ satisfies $\lim_{t\to\infty}\phi(t) = 0$. The Karamata Representation Theorem shows that $NRV_q \subset RV_q$.

If $f \in NRV_{\rho+1}$ $(\rho > 0)$ satisfies (A) and $b \equiv 0$ on $\partial\Omega$ such that $b(x) = k^2(d)(1 + o(1))$ as $d(x) \to 0$, for some $k \in \mathcal{K}$, then for any $a \in (-\infty, \lambda_{\infty,1})$, there is a unique positive solution u_a of Eq. (1). Note that the Keller–Osserman condition is automatically fulfilled. Moreover, we have $\lim_{u\to\infty}\Xi(u) = \lim_{u\to\infty}[F(u)]^{1/2}[f(u)\int_u^\infty (F(s))^{-1/2}\,ds]^{-1} = \rho[2(\rho+2)]^{-1}$ (see [1]).

We have seen in [1] that the uniqueness of u_a is essentially based on the same boundary behaviour shown by any positive solution of (1). The purpose of this Note is to refine the blow-up rate of u_a near $\partial\Omega$ by giving the second term in the expansion of u_a near $\partial\Omega$. This is a more subtle question which represents the goal of more recent literature (see [4] and the references therein). The approach we give is very general and, as a novelty, it relies on the theory of regular variation instituted in the 30's by Karamata and subsequently developed by himself and many others (see [5, 8]). For any $\zeta > 0$, set $\mathcal{K}_{0,\zeta}$ the subset of $\mathcal{K}$ with $\ell_1 = 0$ and $\lim_{t\searrow 0} t^{-\zeta}(\int_0^t k(s)\,ds/k(t))' := L_\star \in \mathbf{R}$. It can be proven that $\mathcal{K}_{0,\zeta} \equiv \mathcal{R}_{0,\zeta}$, where

$$\mathcal{R}_{0,\zeta} = \left\{ \begin{array}{ll} k: & k(u^{-1}) = d_0 u\,[\Lambda(u)]^{-1} \exp\left[-\int_{d_1}^u (s\Lambda(s))^{-1}\,ds\right] \ (u \geq d_1),\ 0 < \Lambda \in C^1[d_1,\infty), \\ & \lim_{u\to\infty}\Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0,\ \lim_{u\to\infty} u^{\zeta+1}\Lambda'(u) = \ell_\star \in \mathbf{R},\ d_0,\ d_1 > 0 \end{array} \right\}.$$

Moreover, $\ell_\star$ and $L_\star$ are connected by $L_\star = -(1+\zeta)\ell_\star/\zeta$ (see [3] for details). Define

$$\mathcal{F}_{\rho\eta} = \{f \in NRV_{\rho+1}\ (\rho > 0) : \ \phi \in RV_\eta \text{ or } -\phi \in RV_\eta\},\ \eta \in (-\rho-2, 0];$$
$$\mathcal{F}_{\rho 0,\tau} = \{f \in \mathcal{F}_{\rho 0} : \ \lim_{u\to\infty}(\ln u)^\tau \phi(u) = \ell^\star \in \mathbf{R}\},\ \tau \in (0,\infty).$$

Our main result establishes the following asymptotic estimate.

THEOREM 1. - *Assume that*

$$b(x) = k^2(d)(1 + \tilde{c}d^\theta + o(d^\theta)) \ \text{ if } d(x) \to 0, \quad \text{where } k \in \mathcal{R}_{0,\zeta},\ \theta > 0,\ \tilde{c} \in \mathbf{R}. \tag{2}$$

Suppose that f fulfills (A) and one of the following growth conditions at infinity:
(i) *$f(u) = Cu^{\rho+1}$ in a neighbourhood of infinity;*

(ii) $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta \neq 0$;
(iii) $f \in \mathcal{F}_{\rho 0,\tau_1}$ *with* $\tau_1 = \varpi/\zeta$, *where* $\varpi = \min\{\theta, \zeta\}$.
Then, for any $a \in (-\infty, \lambda_{\infty,1})$, *the unique positive solution* u_a *of (1) satisfies*

$$u_a(x) = \xi_0 h(d)(1 + \chi d^{\varpi} + o(d^{\varpi})) \quad \text{if } d(x) \to 0, \quad \text{where } \xi_0 = [2(2+\rho)^{-1}]^{1/\rho} \tag{3}$$

and h *is defined by* $\int_{h(t)}^{\infty} [2F(s)]^{-1/2} ds = \int_0^t k(s)\, ds$, *for* $t > 0$ *small enough. The expression of* χ *is*

$$\chi = \begin{cases} -(1+\zeta)\ell_\star (2\zeta)^{-1} \operatorname{Heaviside}(\theta - \zeta) - \tilde{c}\rho^{-1} \operatorname{Heaviside}(\zeta - \theta) := \chi_1 & \text{if (i) or (ii) holds} \\ \chi_1 - \ell^\star \rho^{-1} (-\rho \ell_\star/2)^{\tau_1} [1/(\rho+2) + \ln \xi_0] & \text{if } f \text{ obeys (iii).} \end{cases}$$

Note that the only case related, in same way, to our Theorem 1 corresponds to $\Omega_0 = \emptyset$, $f(u) = u^{\rho+1}$ on $[0, \infty)$, $k(t) = ct^\alpha \in \mathcal{K}$ (where $c, \alpha > 0$), $\theta = 1$ in (2), being studied in [4]. There, the two-term asymptotic expansion of u_a near $\partial\Omega$ ($a \in \mathbf{R}$ since $\lambda_{\infty,1} = \infty$) involves both the distance function $d(x)$ and the mean curvature H of $\partial\Omega$. However, the blow-up rate of u_a we present in Theorem 1 is of a different nature since the class $\mathcal{R}_{0,\zeta}$ does not include $k(t) = ct^\alpha$.

Our main result contributes to the knowledge in some new directions. More precisely, the blow-up rate of the unique positive solution u_a of (1) (found in [1]) is here refined

(a) on the maximal interval $(-\infty, \lambda_{\infty,1})$ for the parameter a, which is in connection with an appropriate semilinear eigenvalue problem; thus, the condition $b > 0$ in Ω (which appears in [4]) is removed by defining the set Ω_0, but we maintain $b \equiv 0$ on $\partial\Omega$ since this is a *natural* restriction inherited from the logistic problem (see [4] for details).

(b) when b satisfies (2), where θ is *any* positive number and k belongs to a very rich class of functions, namely $\mathcal{R}_{0,\zeta}$. The equivalence $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$ shows the connection to the larger class $\mathcal{K}$ (introduced in [1]) for which the uniqueness of u_a holds. In addition, the explicit form of $k \in \mathcal{R}_{0,\zeta}$ shows us how to built $k \in \mathcal{K}_{0,\zeta}$.

(c) for a wide class of functions $f \in NRV_{\rho+1}$ where either $\phi \equiv 0$ (case (i)) or ϕ (resp., $-\phi$) belongs to RV_η with $\eta \in (-\rho-2, 0]$ (cases (ii) and (iii)). Therefore, the theory of regular variation plays a key role in understanding the general framework and the approach as well.

Proof of Theorem 1. - We first state two auxiliary results (see [3] for their proofs).

LEMMA 1. - *Assume (2) and* $f \in NRV_{\rho+1}$ *satisfies* (A). *Then* h *has the following properties:*
(i) $h \in C^2(0,\nu)$, $\lim_{t\searrow 0} h(t) = \infty$ *and* $\lim_{t\searrow 0} h'(t) = -\infty$;
(ii) $\lim_{t\searrow 0} h''(t)/[k^2(t) f(h(t)\xi)] = (2+\rho\ell_1)/[\xi^{\rho+1}(2+\rho)]$, $\forall \xi > 0$;
(iii) $\lim_{t\searrow 0} h(t)/h''(t) = \lim_{t\searrow 0} h'(t)/h''(t) = \lim_{t\searrow 0} h(t)/h'(t) = 0$;
(iv) $\lim_{t\searrow 0} h'(t)/[th''(t)] = -\rho\ell_1/(2+\rho\ell_1)$ *and* $\lim_{t\searrow 0} h(t)/[t^2 h''(t)] = \rho^2\ell_1^2/[2(2+\rho\ell_1)]$;
(v) $\lim_{t\searrow 0} h(t)/[th'(t)] = \lim_{t\searrow 0} [\ln t]/[\ln h(t)] = -\rho\ell_1/2$;
(vi) *If* $\ell_1 = 0$, *then* $\lim_{t\searrow 0} t^j h(t) = \infty$, *for all* $j > 0$;
(vii) $\lim_{t\searrow 0} 1/[t^\zeta \ln h(t)] = -\rho\ell_\star/2$ *and* $\lim_{t\searrow 0} h'(t)/[t^{\zeta+1} h''(t)] = \rho\ell_\star/(2\zeta)$, $\forall k \in \mathcal{R}_{0,\zeta}$.

Let $\tau > 0$ be arbitrary. For any $u > 0$, define $T_{1,\tau}(u) = \{\rho/[2(\rho+2)] - \Xi(u)\}(\ln u)^\tau$ and $T_{2,\tau}(u) = \{f(\xi_0 u)/[\xi_0 f(u)] - \xi_0^\rho\}(\ln u)^\tau$. Note that if $f(u) = Cu^{\rho+1}$, for u in a neighbourhood V_∞ of infinity, then $T_{1,\tau}(u) = T_{2,\tau}(u) = 0$ for each $u \in V_\infty$.

LEMMA 2. - *Assume* (A) *and* $f \in \mathcal{F}_{\rho\eta}$. *The following hold:*
(i) *If* $f \in \mathcal{F}_{\rho 0,\tau}$, *then* $\lim_{u\to\infty} T_{1,\tau}(u) = -\ell^\star/(\rho+2)^2$ *and* $\lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \ell^\star \ln \xi_0$.
(ii) *If* $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta \neq 0$, *then* $\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} T_{2,\tau}(u) = 0$.

Fix $\varepsilon \in (0,1/2)$. We can find $\delta > 0$ such that $d(x)$ is of class C^2 on $\{x \in \mathbf{R}^N : \ d(x) < \delta\}$, k is nondecreasing on $(0,\delta)$, and $h'(t) < 0 < h''(t)$ for all $t \in (0,\delta)$ (see [1] for details). A straightforward computation shows that $\lim_{t\searrow 0} t^{1-\theta}k'(t)/k(t) = \infty$, for every $\theta > 0$. Using now (2), it follows that we can diminish $\delta > 0$ such that $k^2(t)\left[1 + (\tilde{c} - \varepsilon)t^\theta\right]$ is increasing on $(0,\delta)$ and

$$1 + (\tilde{c} - \varepsilon)d^\theta < b(x)/k^2(d) < 1 + (\tilde{c} + \varepsilon)d^\theta, \quad \forall x \in \Omega \text{ with } d \in (0,\delta). \tag{4}$$

We define $u^\pm(x) = \xi_0 h(d)(1 + \chi_\varepsilon^\pm d^\varpi)$, with $d \in (0,\delta)$, where $\chi_\varepsilon^\pm = \chi \pm \varepsilon\,[1 + \text{Heaviside}\,(\zeta - \theta)]/\rho$. Take $\delta > 0$ small enough such that $u^\pm(x) > 0$, for each $x \in \Omega$ with $d \in (0,\delta)$. By the Lagrange mean value theorem, we obtain $f(u^\pm(x)) = f(\xi_0 h(d)) + \xi_0 \chi_\varepsilon^\pm d^\varpi h(d) f'(\Upsilon^\pm(d))$, where $\Upsilon^\pm(d) = \xi_0 h(d)(1 + \lambda^\pm(d)\chi_\varepsilon^\pm d^\varpi)$, for some $\lambda^\pm(d) \in [0,1]$. We claim that

$$\lim_{d\searrow 0} f(\Upsilon^\pm(d))/f(\xi_0 h(d)) = 1. \tag{5}$$

Fix $\sigma \in (0,1)$ and $M > 0$ such that $|\chi_\varepsilon^\pm| < M$. Choose $\mu^\star > 0$ so that $|(1 \pm Mt)^{\rho+1} - 1| < \sigma/2$, for all $t \in (0, 2\mu^\star)$. Let $\mu_\star \in (0, (\mu^\star)^{1/\varpi})$ be such that, for every $x \in \Omega$ with $d \in (0,\mu_\star)$

$$\left| f(\xi_0 h(d)(1 \pm M\mu^\star))/f(\xi_0 h(d)) - (1 \pm M\mu^\star)^{\rho+1} \right| < \sigma/2.$$

Hence, $1 - \sigma < (1 - M\mu^\star)^{\rho+1} - \sigma/2 < f(\Upsilon^\pm(d))/f(\xi_0 h(d)) < (1 + M\mu^\star)^{\rho+1} + \sigma/2 < 1 + \sigma$, for every $x \in \Omega$ with $d \in (0,\mu_\star)$. This proves (5).

Step 1. There exists $\delta_1 \in (0,\delta)$ so that $\Delta u^+ + au^+ - k^2(d)[1 + (\tilde{c} - \varepsilon)d^\theta]f(u^+) \le 0$, $\forall x \in \Omega$ with $d \in (0,\delta_1)$ and $\Delta u^- + au^- - k^2(d)[1 + (\tilde{c} + \varepsilon)d^\theta]f(u^-) \ge 0$, $\forall x \in \Omega$ with $d \in (0,\delta_1)$.

Indeed, for every $x \in \Omega$ with $d \in (0,\delta)$, we have

$$\begin{aligned} &\Delta u^\pm + au^\pm - k^2(d)\left[1 + (\tilde{c} \mp \varepsilon)d^\theta\right] f(u^\pm) \\ = \xi_0 d^\varpi h''(d) \quad &\Big[a\chi_\varepsilon^\pm \tfrac{h(d)}{h''(d)} + \chi_\varepsilon^\pm \Delta d\, \tfrac{h'(d)}{h''(d)} + 2\varpi\chi_\varepsilon^\pm \tfrac{h'(d)}{dh''(d)} + \varpi\chi_\varepsilon^\pm \Delta d\, \tfrac{h(d)}{dh''(d)} \\ &+\varpi(\varpi-1)\chi_\varepsilon^\pm \tfrac{h(d)}{d^2 h''(d)} + \Delta d\, \tfrac{h'(d)}{d^\varpi h''(d)} + \tfrac{a\,h(d)}{d^\varpi h''(d)} + \textstyle\sum_{j=1}^4 \mathcal{S}_j^\pm(d)\Big] \end{aligned} \tag{6}$$

where, for any $t \in (0,\delta)$, we denote

$$\mathcal{S}_1^\pm(t) = (-\tilde{c} \pm \varepsilon)t^{\theta-\varpi}\, k^2(t) f(\xi_0 h(t))/[\xi_0 h''(t)],\ \ \mathcal{S}_2^\pm(t) = \chi_\varepsilon^\pm(1 - k^2(t)h(t)f'(\Upsilon^\pm(t))/h''(t)),$$
$$\mathcal{S}_3^\pm(t) = (-\tilde{c} \pm \varepsilon)\chi_\varepsilon^\pm t^\theta\, k^2(t)h(t)f'(\Upsilon^\pm(t))/h''(t),\ \ \mathcal{S}_4^\pm(t) = t^{-\varpi}(1 - k^2(t)f(\xi_0 h(t))/[\xi_0 h''(t)]).$$

By Lemma 1 (ii), we find $\lim_{t\searrow 0} k^2(t) f(\xi_0 h(t))[\xi_0 h''(t)]^{-1} = 1$, which yields $\lim_{t\searrow 0} \mathcal{S}_1^\pm(t) = (-\tilde{c} \pm \varepsilon)\text{Heaviside}\,(\zeta - \theta)$. Using [1, Lemma 1] and (5), we obtain $\lim_{t\searrow 0} k^2(t)h(t)f'(\Upsilon^\pm(t))/h''(t) = \rho + 1$. Hence, $\lim_{t\searrow 0} \mathcal{S}_2^\pm(t) = -\rho\chi_\varepsilon^\pm$ and $\lim_{t\searrow 0} \mathcal{S}_3^\pm(t) = 0$.

Using the expression of h'', we derive $\mathcal{S}_4^\pm(t) = \frac{k^2(t)f(h(t))}{h''(t)} \sum_{i=1}^3 \mathcal{S}_{4,i}(t)$, $\forall t \in (0,\delta)$, where we denote $\mathcal{S}_{4,1}(t) = 2\frac{\Xi(h(t))}{t^\varpi}(\int_0^t k(s)\,ds/k(t))'$, $\mathcal{S}_{4,2}(t) = 2\frac{T_{1,\tau_1}(h(t))}{[t^\zeta \ln h(t)]^{\tau_1}}$ and $\mathcal{S}_{4,3}(t) = -\frac{T_{2,\tau_1}(h(t))}{[t^\zeta \ln h(t)]^{\tau_1}}$.

Since $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$, we find $\lim_{t\searrow 0} \mathcal{S}_{4,1}(t) = -(1+\zeta)\rho\ell_\star\zeta^{-1}(\rho+2)^{-1}\,\text{Heaviside}\,(\theta - \zeta)$.

Cases (i), (ii). By Lemma 1 (vii) and Lemma 2 (ii), we find $\lim_{t\searrow 0} \mathcal{S}_{4,2}(t) = \lim_{t\searrow 0} \mathcal{S}_{4,3}(t) = 0$. In view of Lemma 1 (ii), we derive that $\lim_{t\searrow 0} \mathcal{S}_4^\pm(t) = -(1+\zeta)\rho\ell_\star(2\zeta)^{-1}\,\text{Heaviside}\,(\theta - \zeta)$.

Case (iii). By Lemma 1 (vii) and Lemma 2 (i), $\lim_{t\searrow 0} \mathcal{S}_{4,2}(t) = -2\ell^\star(\rho+2)^{-2}(-\rho\ell_\star/2)^{\tau_1}$ and $\lim_{t\searrow 0} \mathcal{S}_{4,3}(t) = -2\ell^\star(\rho+2)^{-1}(-\rho\ell_\star/2)^{\tau_1}\ln\xi_0$. Using Lemma 1 (ii) once more, we arrive at $\lim_{t\searrow 0} \mathcal{S}_4^\pm(t) = -(1+\zeta)\rho\ell_\star(2\zeta)^{-1}\,\text{Heaviside}\,(\theta - \zeta) - \ell^\star(-\rho\ell_\star/2)^{\tau_1}[1/(\rho+2) + \ln\xi_0]$.

Note that in each of the cases (i)–(iii), the definition of $\chi_\varepsilon^\pm$ yields $\lim_{t\searrow 0} \sum_{j=1}^4 \mathcal{S}_j^+(t) = -\varepsilon < 0$ and $\lim_{t\searrow 0} \sum_{j=1}^4 \mathcal{S}_j^-(t) = \varepsilon > 0$. By Lemma 1 (vii), $\lim_{t\searrow 0} \frac{h'(t)}{t^\varpi h''(t)} = 0$. But $\lim_{t\searrow 0} \frac{h(t)}{h'(t)} = 0$, so $\lim_{t\searrow 0} \frac{h(t)}{t^\varpi h''(t)} = 0$. Thus, using Lemma 1 [(iii), (iv)], relation (6) concludes our Step 1.

Step 2. There exists M^+, $\delta^+ > 0$ such that $u_a(x) \le u^+(x) + M^+$, for all $x \in \Omega$ with $0 < d < \delta^+$.

Define $(0,\infty) \ni u \longmapsto \Psi_x(u) = au - b(x)f(u)$, $\forall x$ with $d \in (0,\delta_1)$. Clearly, $\Psi_x(u)$ is decreasing when $a \le 0$. Suppose $a \in (0, \lambda_{\infty,1})$. Obviously, $f(t)/t : (0,\infty) \to (f'(0),\infty)$ is bijective. Let $\delta_2 \in (0,\delta_1)$ be such that $b(x) < 1$, $\forall x$ with $d \in (0,\delta_2)$. Let u_x define the unique positive solution of $b(x)f(u)/u = a + f'(0)$, $\forall x$ with $d \in (0,\delta_2)$. Hence, for any x with $d \in (0,\delta_2)$, $u \to \Psi_x(u)$ is decreasing on (u_x,∞). But $\lim_{d(x)\searrow 0} \frac{b(x)f(u^+(x))}{u^+(x)} = +\infty$ (use $\lim_{d(x)\searrow 0} u^+(x)/h(d) = \xi_0$, (A) and Lemma 1 [(ii) and (iii)]). So, for δ_2 small enough, $u^+(x) > u_x$, $\forall x$ with $d \in (0,\delta_2)$.

Fix $\sigma \in (0,\delta_2/4)$ and set $\mathcal{N}_\sigma := \{x \in \Omega :\ \sigma < d(x) < \delta_2/2\}$. We define $u_\sigma^*(x) = u^+(d-\sigma,s) + M^+$, where (d,s) are the local coordinates of $x \in \mathcal{N}_\sigma$. We choose $M^+ > 0$ large enough to have $u_\sigma^*(\delta_2/2,s) \ge u_a(\delta_2/2,s)$, $\forall \sigma \in (0,\delta_2/4)$ and $\forall s \in \partial\Omega$. Using (4) and Step 1, we find

$$\begin{aligned}
-\Delta u_\sigma^*(x) &\ge au^+(d-\sigma,s) - [1+(\tilde{c}-\varepsilon)(d-\sigma)^\theta]k^2(d-\sigma)f(u^+(d-\sigma,s)) \\
&\ge au^+(d-\sigma,s) - [1+(\tilde{c}-\varepsilon)d^\theta]k^2(d)f(u^+(d-\sigma,s)) \ge \Psi_x(u^+(d-\sigma,s)) \\
&\ge \Psi_x(u_\sigma^*) = au_\sigma^*(x) - b(x)f(u_\sigma^*(x)) \quad \text{in } \mathcal{N}_\sigma.
\end{aligned}$$

Thus, by [2, Lemma 1], $u_a \le u_\sigma^*$ in $\mathcal{N}_\sigma$, $\forall \sigma \in (0,\delta_2/4)$. Letting $\sigma \to 0$, we have proved Step 2.

Step 3. There exists M^-, $\delta^- > 0$ such that $u_a(x) \ge u^-(x) - M^-$, for all $x \in \Omega$ with $0 < d < \delta^-$.

For every $r \in (0,\delta)$, define $\Omega_r = \{x \in \Omega :\ 0 < d(x) < r\}$. We will prove that for $\lambda > 0$ sufficiently small, $\lambda u^-(x) \le u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$. Indeed, fix arbitrarily $\sigma \in (0,\delta_2/4)$. Define $v_\sigma^*(x) = \lambda u^-(d+\sigma,s)$, for $x = (d,s) \in \Omega_{\delta_2/2}$. We choose $\lambda \in (0,1)$ small enough such that $v_\sigma^*(\delta_2/4,s) \le u_a(\delta_2/4,s)$, $\forall \sigma \in (0,\delta_2/4)$, $\forall s \in \partial\Omega$. Using (4), Step 1 and (A), we find

$$\begin{aligned}
\Delta v_\sigma^*(x) + av_\sigma^*(x) &\ge \lambda k^2(d+\sigma)[1+(\tilde{c}+\varepsilon)(d+\sigma)^\theta]f(u^-(d+\sigma,s)) \\
&\ge k^2(d)[1+(\tilde{c}+\varepsilon)d^\theta]f(\lambda u^-(d+\sigma,s)) \ge bf(v_\sigma^*),
\end{aligned}$$

for all $x = (d,s) \in \Omega_{\delta_2/4}$, that is v_σ^* is a subsolution of $\Delta u + au = b(x)f(u)$ in $\Omega_{\delta_2/4}$. By [2, Lemma 1], we conclude that $v_\sigma^* \le u_a$ in $\Omega_{\delta_2/4}$. Letting $\sigma \to 0$, we find $\lambda u^-(x) \le u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$.

Since $\lim_{d\searrow 0} u^-(x)/h(d) = \xi_0$, by using (A) and Lemma 1 [(ii), (iii)], we can easily obtain $\lim_{d\searrow 0} k^2(d)f(\lambda^2 u^-(x))/u^-(x) = \infty$. So, there exists $\tilde{\delta} \in (0,\delta_2/4)$ such that

$$k^2(d)[1+(\tilde{c}+\varepsilon)d^\theta]f(\lambda^2 u^-)/u^- \ge \lambda^2|a|, \quad \forall x \in \Omega \text{ with } 0 < d \le \tilde{\delta}. \tag{7}$$

By Lemma 1 [(i) and (v)], we deduce that $u^-(x)$ decreases with d when $d \in (0,\tilde{\delta})$ (if necessary, $\tilde{\delta} > 0$ is diminished). Choose $\delta_* \in (0,\tilde{\delta})$, close enough to $\tilde{\delta}$, such that

$$h(\delta_*)(1+\chi_\varepsilon^-\delta_*^\varpi)/[h(\tilde{\delta})(1+\chi_\varepsilon^-\tilde{\delta}^\varpi)] < 1+\lambda. \tag{8}$$

For each $\sigma \in (0,\tilde{\delta}-\delta_*)$, we define $z_\sigma(x) = u^-(d+\sigma,s) - (1-\lambda)u^-(\delta_*,s)$. We prove that z_σ is a subsolution of $\Delta u + au = b(x)f(u)$ in Ω_{δ_*}. Using (8), $z_\sigma(x) \ge u^-(\tilde{\delta},s) - (1-\lambda)u^-(\delta_*,s) > 0$ $\forall x = (d,s) \in \Omega_{\delta_*}$. By (4) and Step 1, z_σ is a subsolution of $\Delta u + au = b(x)f(u)$ in Ω_{δ_*} if

$$k^2(d+\sigma)[1+(\tilde{c}+\varepsilon)(d+\sigma)^\theta]\left[f(u^-(d+\sigma,s)) - f(z_\sigma(d,s))\right] \ge a(1-\lambda)u^-(\delta_*,s), \tag{9}$$

for all $(d,s) \in \Omega_{\delta_*}$. Applying the Lagrange mean value theorem and (A), we infer that (9) is a consequence of $k^2(d+\sigma)[1+(\tilde{c}+\varepsilon)(d+\sigma)^\theta]f(z_\sigma(d,s))/z_\sigma(d,s) \ge |a|$, $\forall (d,s) \in \Omega_{\delta_*}$. This inequality holds by virtue of (7), (8) and the decreasing character of u^- with d.

On the other hand, $z_\sigma(\delta_*,s) \le \lambda u^-(\delta_*,s) \le u_a(x)$, $\forall x = (\delta_*,s) \in \Omega$. Clearly, $\limsup_{d\to 0}(z_\sigma - u_a)(x) = -\infty$ and $b > 0$ in Ω_{δ_*}. Thus, by [2, Lemma 1], $z_\sigma \le u_a$ in Ω_{δ_*}, $\forall \sigma \in (0,\tilde{\delta}-\delta_*)$. Letting $\sigma \to 0$, we conclude the assertion of Step 3.

By Steps 2 and 3, $\chi_\varepsilon^+ \geq \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} - M^+/[\xi_0 d^{\varpi} h(d)]$ $\forall x \in \Omega$ with $d \in (0, \delta^+)$ and $\chi_\varepsilon^- \leq \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} + M^-/[\xi_0 d^{\varpi} h(d)]$ $\forall x \in \Omega$ with $d \in (0, \delta^-)$. Passing to the limit as $d \to 0$ and using Lemma 1 (vi), we obtain $\chi_\varepsilon^- \leq \liminf_{d\to 0}\{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi}$ and $\limsup_{d\to 0}\{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} \leq \chi_\varepsilon^+$. Letting $\varepsilon \to 0$, we conclude our proof. ∎

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Existence implies uniqueness for a class of singular anisotropic elliptic boundary value problems

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We consider a singular anisotropic quasilinear problem with Dirichlet boundary condition and we establish two sufficient conditions for the uniqueness of the solution, provided such a solution exists. The proofs use elementary tools and they are based on a general comparison lemma combined with the generalized mean value theorem.

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1 Introduction and the main results

Singular anisotropic boundary value problems arise naturally when studying many concrete situations. We refer to Čanić-Keyfitz [1] for the study of self-similar solutions of conservation laws in two dimensions. We also mention Ding-Liu [5] where it is studied another anisotropic problem in the plane. Their model is closely related to the phase transition problem in anisotropic superconductivity with "thermal noise" term.

In [2], Choi, Lazer and McKenna studied a problem that is linked to an equation arising in fluid dynamics. They proved that the singular elliptic boundary value problem

$$\begin{cases} u^a u_{xx} + u^b u_{yy} + p(x,y) = 0, & (x,y) \in \Omega \\ u = 0, & (x,y) \in \partial\Omega \end{cases} \tag{1}$$

has a positive classical solution, where $\Omega \subset \mathbf{R}^2$ is a bounded convex domain with smooth boundary, p is a positive Hölder continuous function and the constants a, b satisfy $a > b \geq 0$. Choi, Lazer and McKenna also developed a new comparison principle for quasilinear problems that is based on the method of sub- and super-solutions.

Recently Choi and McKenna [3] removed the assumption that the dimension be restricted to two, but they also retained the convexity assumption which is crucial in the construction of a super-solution ψ, satisfying the boundary conditions. More exactly, they showed that the boundary value problem

$$\begin{cases} \displaystyle\sum_{i=1}^{N} u^{a_i} u_{x_i x_i} + p(x) = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2}$$

has at least one positive classical solution u, such that $u(x) \leq \psi(x)$ for all $x \in \Omega$, where $\Omega \subset \mathbf{R}^N$ ($N \geq 1$) is a bounded convex domain with smooth boundary and $a_1 \geq a_2 \geq \cdots \geq a_N \geq 0$, with $a_1 > a_N$. Choi and McKenna point out that the most significant omission of their paper is the absence of any information on the uniqueness of solutions. In this direction there are known very few results which hold only for the two dimensional case.

Lair and Shaker proved in [7] a uniqueness result related to (1) and they required neither the domain Ω to be convex nor the function p to be as smooth as in [2]. They made only the assumption that there is some solution u for which u_{xx} is bounded above appropriately. In their paper there are distinguished two different situations: $a-b \geq 1$, resp., $a-b<1$.

Reichel [8] established that problem (1) has at most one positive classical solution. It is assumed that

$$p(\tau_1 x, \tau_2 y) \geq p(x,y) \quad \text{for all } (x,y) \in \Omega,\ \tau_i \in [0,1]$$

and the bounded domain Ω (with $0 \in \Omega$) satisfies an interior rectangle condition, i.e., for each $(x,y) \in \partial\Omega$ the rectangle $\{(\tau_1 x, \tau_2 y):\ \tau_i \in [0,1)\}$ is a subset of Ω.

It is natural to ask us if it is possible to give a uniqueness result which holds for more general degenerate quasilinear operators and for a larger class of functions p, with no assumption on the geometry of the domain or the dimension of the space.

For this aim, we consider the singular anisotropic elliptic boundary value problem

$$\begin{cases} \displaystyle\sum_{i=1}^{N-1} f_i(u)\, u_{x_i x_i} + u_{yy} + p(x)\, g(u) = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $N \geq 2$ and p is a positive continuous function on $\overline{\Omega}$. We have denoted the last coordinate x_N by y and we shall use notation x' for the first $(N-1)$ coordinates.

Throughout this paper, we assume that the following hypotheses are fulfilled

($\mathbf{H}_1$) $\quad f_i, g : (0,\infty) \to (0,\infty)$, $i = \overline{1, N-1}$ are C^1-functions;

($\mathbf{H}_2$) $\quad f_i$, $i = \overline{1, N-1}$ is nondecreasing on $(0,\infty)$ and g is nonincreasing on $(0,\infty)$.

Since Ω is bounded, we can make a translation of the domain so that it lies in the interior of the strip $\mathbf{R}^{N-1} \times [0,\ell]$ for some $\ell > 0$. The fact that $p \in C(\overline{\Omega})$ is a positive function implies the existence of $\alpha \geq 0$ and $\beta > 0$ such that $p(x) \in [\alpha, \beta]$ for each $x \in \overline{\Omega}$.

Set

$$D = \{y \in [0,\ell]:\ \exists x' \text{ such that } (x',y) \in \overline{\Omega}\}.$$

We can suppose, without loss of generality, that $\ell \notin D$.

Let ψ be the unique positive function defined by

$$\int_0^{\psi(y)} \frac{1}{g(t)}\, dt = \frac{\beta}{2}(\ell y - y^2), \quad \text{for any } y \in [0,\ell]. \tag{4}$$

It is obvious that

$$\max_{y\in D}\psi(y)\le\max_{y\in[0,\ell]}\psi(y)=A, \tag{5}$$

where $A>0$ is uniquely defined by

$$\int_0^A\frac{1}{g(t)}\,dt=\frac{\beta}{8}\ell^2. \tag{6}$$

We also assume

$(\mathbf{H}_3)$ $f_1'>0$ on $(0,A]$.

In the first result of this paper we impose the condition

$(\mathbf{C}_1)$ there exists and is finite $\lim\limits_{x\searrow 0}\dfrac{f_1 f_i'}{f_1'}(x)$, for all $i=\overline{2,N-1}$.

In view of this hypothesis we observe that for any $i=\overline{2,N-1}$ it makes sense to define

$$m_i=\min_{[0,A]}\frac{\left(\frac{f_i}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}=\min_{[0,A]}\left(f_i-\frac{f_i'f_1}{f_1'}\right)\quad\text{and}\quad M_i=\max_{[0,A]}\frac{\left(\frac{f_i}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}=\max_{[0,A]}\left(f_i-\frac{f_i'f_1}{f_1'}\right).$$

For any $x\in\Omega$ we define the sets

$$P_x=\{2\le i\le N-1;\ u_{x_ix_i}(x)\ge 0\}\quad\text{and}\quad N_x=\{2\le i\le N-1;\ u_{x_ix_i}(x)<0\}.$$

Our first result asserts that the existence of a positive solution $u\in C^{2,\alpha}(\Omega)\cap C(\overline{\Omega})$ of (3) ensures its uniqueness, provided that the expression $\sum_{i\in P_x}m_iu_{x_ix_i}+\sum_{i\in N_x}M_iu_{x_ix_i}+u_{yy}$ is bounded below appropriately.

Theorem 1 *Assume* $(\mathbf{H}_1)$-$(\mathbf{H}_3)$ *and* $(\mathbf{C}_1)$ *hold. There exists a positive constant* K_1*, depending on* f_1*,* g*,* p *and* Ω*, such that if* u *is a positive solution of (3) satisfying*

$$\sum_{i\in P_x}m_iu_{x_ix_i}+\sum_{i\in N_x}M_iu_{x_ix_i}+u_{yy}>-K_1\quad\text{in }\Omega \tag{7}$$

then u *is the unique solution of (3).*

We now drop the assumption $(\mathbf{C}_1)$ but we require

$(\mathbf{C}_2)$ $\dfrac{f_i}{f_1}$, $i=\overline{2,N-1}$ is nonincreasing on $(0,\infty)$.

Our next theorem shows that the uniqueness of solution to (3) is assured if we find a positive solution $u\in C^{2,\alpha}(\Omega)\cap C(\overline{\Omega})$ with the property that $u_{x_1x_1}+\sum_{i\in P_x}\frac{f_i(u)}{f_1(u)}u_{x_ix_i}+\sum_{i\in N_x}\left(\inf_{(0,A)}\frac{f_i'}{f_1'}\right)u_{x_ix_i}$ is bounded above appropriately.

Theorem 2 *Assume* $(\mathbf{H}_1)$-$(\mathbf{H}_3)$ *and* $(\mathbf{C}_2)$ *hold. There exists a non-negative constant* K_2 *depending on* f_1*,* g*,* p *and* Ω*, such that if* u *is a positive solution of problem (3) satisfying*

$$u_{x_1x_1}+\sum_{i\in P_x}\frac{f_i(u)}{f_1(u)}u_{x_ix_i}+\sum_{i\in N_x}\left(\inf_{(0,A)}\frac{f_i'}{f_1'}\right)u_{x_ix_i}<K_2\quad\text{in }\Omega \tag{8}$$

then u *is the unique solution of (3).*

We point out that hypotheses (7) and (8) should be understood as sufficient conditions that guarantee the uniqueness of the solution, provided such a solution exists. Problems related to uniqueness for singular anisotropic quasilinear boundary value problems have been recently studied by Hill, Moore and Reichel in [6]. In [6] the authors impose a topological contraint to the boundary and the proof of the uniqueness of the solution uses essentially the fact that Ω satisfies a weighted starshapedness condition. In order to illustrate our above stated results, let us consider the problem

$$\begin{cases} \sum_{i=1}^{N} u^{a_i} u_{x_i x_i} + 2\sum_{i=1}^{N} (1-|x|^2)^{a_i} = 0, & \text{if } x \in B(0,1) \subset \mathbf{R}^N \\ u = 0, & \text{if } |x| = 1, \end{cases} \tag{9}$$

where $a_1 \geq \cdots \geq a_N \geq 0$ and $a_1 > a_N$. By Theorem 4.3 in [6], this problem has a unique solution. The same conclusion follows from our results. Indeed, let us observe that the functions $f_i(t) = t^{a_i - a_N}$ and $g(t) = t^{-a_N}$ fulfill conditions $(\mathbf{H}_1)$-$(\mathbf{H}_3)$ and $(\mathbf{C}_1)$, with $\alpha = 0$, $\beta = 2N$, $A = [N(a_N+1)]^{1/(a_N+1)}$, $m_i = 0$ and $M_i = (a_1 - a_i)(a_1 - a_N)^{-1} A^{a_i - a_N}$, for $1 \leq i \leq N-1$. Choosing

$$K_1 > 2 + \frac{2}{a_1 - a_N} \sum_{i=2}^{N-1} (a_1 - a_i) A^{a_i - a_N} \, ,$$

it follows by Theorem 1 that $u(x) = 1 - |x|^2$ is the unique solution of problem (9).

The main difficulty in the treatment of (3) is the lack of the usual comparison principle between sub- and super-solution, due to the anisotropic character of the equation. To this end, using a result of Choi and McKenna, we will state in Section 2 a comparison principle which is suitable for (3).

2 An auxiliary result

In this section we prove that the number A given by (6) is an upper bound for every positive classical solution of problem (3). To this end, we make use of a comparison lemma on a class of quasilinear elliptic equations established in Choi-McKenna [3]. In view of this result we can obtain L^∞ bounds on the solutions to this class of equations using the method of sub- and super-solutions. Consider the problem

$$\begin{cases} \sum_{i=1}^{N-1} f_i(x,u)\, u_{x_i x_i} + u_{yy} + p(x)\, g(x,u) = 0, & \text{in } \Omega \\ u = u_0, & \text{on } \partial\Omega, \end{cases} \tag{10}$$

with $u_0|_{\partial\Omega} \geq 0$, where the functions f_i, g and p satisfy the assumptions

$(\mathbf{A}_1)$ $f_i : \Omega \times [0,\infty) \to [0,\infty)$ is continuous and $f_i(x,\cdot)$ is nondecreasing for each $x \in \Omega$;

$(\mathbf{A}_2)$ $g : \Omega \times (0,\infty) \to (0,\infty)$ is continuous, and $g(x,\cdot)$ is nonincreasing for each $x \in \Omega$;

$(\mathbf{A}_3)$ $p : \overline{\Omega} \to \mathbf{R}$ is continuous, and there exist positive constants α and β such that

$$0 < \alpha \leq p(x) \leq \beta \quad \text{for all } x \in \overline{\Omega}.$$

Assume that

(**L**) There exists a sub-solution $\varphi \in C(\overline{\Omega}) \cap C^2(\Omega)$ with $\varphi > 0$ on Ω satisfying

$$\sum_{i=1}^{N-1} f_i(x,\varphi)\,\varphi_{x_i x_i} + \varphi_{yy} + p(x)\,g(x,\varphi) > 0, \quad \text{in } \Omega$$

$$\varphi_{x_i x_i} \le 0, \quad \text{in } \Omega, \text{ for any } i = 1, 2, \cdots, N-1,$$

and $\varphi \le u_0$ on $\partial\Omega$.

(**U**) There exists a super-solution $\psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ with $\psi > 0$ in Ω satisfying

$$\sum_{i=1}^{N-1} f_i(x,\psi)\,\psi_{x_i x_i} + \psi_{yy} + p(x)\,g(x,\psi) \le 0, \quad \text{in } \Omega$$

$$\psi_{x_i x_i} \le 0, \quad \text{in } \Omega, \text{ for any } i = 1, 2, \cdots, N-1,$$

and $\psi \ge u_0$ on $\partial\Omega$.

Lemma 1 *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_3)$*,* (**L**) *and* (**U**) *hold. Then any positive solution u of (3) satisfies $u \le A$ in $\overline{\Omega}$, where A is defined in (6).*

Proof. Under the above hypotheses, Choi and McKenna proved in [3] that every solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of problem (10), with $u > 0$ in Ω, satisfies

$$\varphi \le u \le \psi \quad \text{in } \overline{\Omega}.$$

Moreover, if only conditions $(\mathbf{A}_1) - (\mathbf{A}_3)$ and (**U**) hold, then $u \le \psi$ in $\overline{\Omega}$.

It is easy to check that the function ψ defined in (4) satisfies condition (**U**) considered for our problem (3). Therefore, by the Choi-McKenna comparison lemma and (5), we find that every positive classical solution of (3) is bounded above by the same number A defined in (6). □

3 Proof of Theorem 1

Let u and v be solutions of (3) and let u satisfy (7), where

$$K_1 = \frac{\pi^2}{\ell^2}\frac{1}{f_1(A)} \inf_{(0,A)} \frac{f_1^2}{f_1'} + \alpha \inf_{(0,A)} \frac{\left(\frac{g}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}.$$

We prove in what follows that $u = v$ in $\overline{\Omega}$. Set

$$w(x) = \frac{u(x',y)}{s(y)}, \quad z(x) = \frac{v(x',y)}{s(y)},$$

where

$$s(y) = \sin\frac{\pi y}{\ell}, \quad c(y) = \cos\frac{\pi y}{\ell} \qquad y \in (0,\ell).$$

Since $s>0$ and $s\in C^\infty$, it follows that w and z are well defined and they are as smooth as u and v respectively on Ω. A simple computation shows that w satisfies the boundary value problem

$$\begin{cases} \sum_{i=1}^{N-1} s f_i(u) w_{x_i x_i} + \frac{2\pi c}{\ell} w_y + s w_{yy} - \frac{\pi^2 s}{\ell^2} w + p(x) g(u) = 0, & \text{in } \Omega \\ w = 0, & \text{on } \partial\Omega\,. \end{cases} \tag{11}$$

Similarly,

$$\begin{cases} \sum_{i=1}^{N-1} s f_i(v) z_{x_i x_i} + \frac{2\pi c}{\ell} z_y + s z_{yy} - \frac{\pi^2 s}{\ell^2} z + p(x) g(v) = 0, & \text{in } \Omega \\ z = 0, & \text{on } \partial\Omega\,. \end{cases} \tag{12}$$

Relations (11) and (12) yield

$$\begin{aligned} &\sum_{i=1}^{N-1} s \frac{f_i(v)}{f_1(v)} (z-w)_{x_i x_i} + \sum_{i=2}^{N-1} s \left[\left(\frac{f_i}{f_1}\right)(v) - \left(\frac{f_i}{f_1}\right)(u) \right] w_{x_i x_i} + \frac{2\pi c}{\ell} \frac{1}{f_1(v)} (z-w)_y + \\ &\frac{2\pi c}{\ell} \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)} \right) w_y + s \frac{1}{f_1(v)} (z-w)_{yy} + s \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)} \right) w_{yy} - \\ &\frac{\pi^2 s}{\ell^2} \frac{1}{f_1(v)} (z-w) - \frac{\pi^2 s}{\ell^2} \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)} \right) w + p(x) \left[\left(\frac{g}{f_1}\right)(v) - \left(\frac{g}{f_1}\right)(u) \right] = 0. \end{aligned}$$

Whenever $z \neq w$ we can rewrite the above equation as follows

$$\sum_{i=1}^{N-1} s \frac{f_i(v)}{f_1(v)} (z-w)_{x_i x_i} + s \frac{1}{f_1(v)} (z-w)_{yy} + \frac{2\pi c}{\ell} \frac{1}{f_1(v)} (z-w)_y + \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)} \right) Q(z,w) = 0 \tag{13}$$

where

$$Q(z,w) = u_{yy} + \sum_{i=2}^{N-1} \frac{\left(\frac{f_i}{f_1}\right)(v) - \left(\frac{f_i}{f_1}\right)(u)}{\left(\frac{1}{f_1}\right)(v) - \left(\frac{1}{f_1}\right)(u)} u_{x_i x_i} - \frac{\pi^2}{\ell^2} \frac{1}{f_1(v)} \frac{v-u}{\left(\frac{1}{f_1}\right)(v) - \left(\frac{1}{f_1}\right)(u)} + p(x) \frac{\left(\frac{g}{f_1}\right)(v) - \left(\frac{g}{f_1}\right)(u)}{\left(\frac{1}{f_1}\right)(v) - \left(\frac{1}{f_1}\right)(u)}.$$

In order to conclude the proof it is enough to show that

$$Q(z,w) > 0 \quad \text{whenever } z \neq w. \tag{14}$$

Indeed, if $(z-w)>0$ at some point in Ω, then $\max_{\overline{\Omega}}(z-w)$ is achieved in Ω, since $z=w=0$ on $\partial\Omega$. At that point we have

$$(z-w)_{x_i x_i} \le 0, \quad (z-w)_{yy} \le 0, \quad (z-w)_y = 0 \quad \text{and} \quad \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)} \right) Q(z,w) < 0$$

which contradicts (13). A similar argument shows that $(z-w)$ cannot be negative at any point in Ω. Hence $z=w$ in Ω which implies $u=v$ on $\overline{\Omega}$.

For every $x\in\Omega$, let us define

$$\mu(x) = \min(u(x), v(x)) \quad \text{and} \quad \nu(x) = \max(u(x), v(x))\,.$$

Thus, by Lemma 1, $\nu \le A$ in Ω.

In (13) we apply the Cauchy generalized mean value theorem on every interval $[\mu(x), \nu(x)]$ where $x \in \Omega$ is taken such that $z(x) \neq w(x)$. Hence, for all $i = \overline{2, N-1}$ we obtain the existence of $\xi_i(x)$, $\sigma(x)$, $\lambda(x) \in (\mu(x), \nu(x)) \subset (0, A)$ such that

$$m_i \le \frac{\left(\frac{f_i}{f_1}\right)(v(x)) - \left(\frac{f_i}{f_1}\right)(u(x))}{\left(\frac{1}{f_1}\right)(v(x)) - \left(\frac{1}{f_1}\right)(u(x))} = \frac{\left(\frac{f_i}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}(\xi_i(x)) \le M_i \tag{15}$$

$$-\frac{v(x) - u(x)}{\left(\frac{1}{f_1}\right)(v(x)) - \left(\frac{1}{f_1}\right)(u(x))} = \frac{f_1^2}{f_1'}(\sigma(x)) \ge \inf_{(0,A)} \frac{f_1^2}{f_1'} \tag{16}$$

$$\frac{\left(\frac{g}{f_1}\right)(v(x)) - \left(\frac{g}{f_1}\right)(u(x))}{\left(\frac{1}{f_1}\right)(v(x)) - \left(\frac{1}{f_1}\right)(u(x))} = \frac{\left(\frac{g}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}(\lambda(x)) \ge \inf_{(0,A)} \frac{\left(\frac{g}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'}. \tag{17}$$

Using (15), (16) and (17) we find

$$Q(z, w) \ge u_{yy} + \sum_{i \in P_x} m_i u_{x_i x_i} + \sum_{i \in N_x} M_i u_{x_i x_i} + \frac{\pi^2}{\ell^2} \frac{1}{f_1(A)} \inf_{(0,A)} \frac{f_1^2}{f_1'} + \alpha \inf_{(0,A)} \frac{\left(\frac{g}{f_1}\right)'}{\left(\frac{1}{f_1}\right)'} =$$
$$u_{yy} + \sum_{i \in P_x} m_i u_{x_i x_i} + \sum_{i \in N_x} M_i u_{x_i x_i} + K_1.$$

Since the solution u satisfies (7) we obtain that relation (14) is true. This completes the proof. □

4 Proof of Theorem 2

Let u and v be two solutions of (3) and set

$$K_2 = -\alpha \sup_{(0,A)} \frac{g'}{f_1'} \ge 0\,. \tag{18}$$

The functions w, z, μ and ν will have the same signification as in the above proof.

By (11) and (12) it follows that

$$\begin{aligned}&\sum_{i=1}^{N-1} s f_i(v)(z-w)_{x_i x_i} + \sum_{i=1}^{N-1} s[f_i(v) - f_i(u)] w_{x_i x_i} + \frac{2\pi c}{\ell}(z-w)_y + s(z-w)_{yy} - \\ &\frac{\pi^2 s}{\ell^2}(z-w) + p(x)[g(v) - g(u)] = 0.\end{aligned} \tag{19}$$

Whenever $z \neq w$, relation (19) may be rewritten in the following form

$$\sum_{i=1}^{N-1} s f_i(v)(z-w)_{x_i x_i} + \frac{2\pi c}{\ell}(z-w)_y + s(z-w)_{yy} + [f_1(v) - f_1(u)] R(z, w) = 0\,,$$

where

$$R(z,w) = u_{x_1x_1} + \sum_{i=2}^{N-1} \frac{f_i(v) - f_i(u)}{f_1(v) - f_1(u)} u_{x_ix_i} - \frac{\pi^2}{\ell^2} \frac{v-u}{f_1(v) - f_1(u)} + p(x) \frac{g(v) - g(u)}{f_1(v) - f_1(u)}.$$

Using the maximum principle (as we did in the proof of Theorem 1) we see that the proof will be concluded if we prove that

$$R(z,w) < 0 \quad \text{whenever } z \neq w.$$

From now on, we shall consider only the points $x \in \Omega$ with the property that $z(x) \neq w(x)$. For these points, we apply again the Cauchy generalized mean value theorem on $[\mu(x), \nu(x)]$ and we obtain $\eta_i(x)$, $\theta(x)$, $\zeta(x) \in (\mu(x), \nu(x)) \subset (0, A)$ such that

$$\frac{f_i(v(x)) - f_i(u(x))}{f_1(v(x)) - f_1(u(x))} = \frac{f_i'}{f_1'}(\eta_i(x)) \geq \inf_{(0,A)} \frac{f_i'}{f_1'}, \quad i = \overline{2, N-1} \tag{20}$$

$$\frac{v(x) - u(x)}{f_1(v(x)) - f_1(u(x))} = \frac{1}{f_1'(\theta(x))} \tag{21}$$

$$\frac{g(v(x)) - g(u(x))}{f_1(v(x)) - f_1(u(x))} = \frac{g'}{f_1'}(\zeta(x)) \leq \sup_{(0,A)} \frac{g'}{f_1'} \leq 0. \tag{22}$$

It is easy to verify that hypothesis $(\mathbf{C}_2)$ implies

$$\frac{f_i(v(x)) - f_i(u(x))}{f_1(v(x)) - f_1(u(x))} \leq \frac{f_i(u(x))}{f_1(u(x))} \quad \text{for all } i = \overline{2, N-1}. \tag{23}$$

On the other hand, since f_1 is increasing on $(0, A)$,

$$\frac{v(x) - u(x)}{f_1(v(x)) - f_1(u(x))} > 0. \tag{24}$$

Combining relations (20), (22), (23) and (24) with the expression of $R(z,w)$ we deduce that

$$R(z,w) < u_{x_1x_1} + \sum_{i \in P_x} \frac{f_i(u)}{f_1(u)} u_{x_ix_i} + \sum_{i \in N_x} \left(\inf_{(0,A)} \frac{f_i'}{f_1'} \right) u_{x_ix_i} + \alpha \sup_{(0,A)} \frac{g'}{f_1'}.$$

Since u is a solution of (3) satisfying (8) we deduce that $R(z,w)$ is negative. This completes our proof. □

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Chapitre III
Problèmes elliptiques non lisses: théories de Clarke et de Degiovanni

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A MULTIPLICITY THEOREM
FOR LOCALLY LIPSCHITZ PERIODIC FUNCTIONALS

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Abstract. We prove in this paper a multiplicity theorem of the Ljusternik-Schnirelmann type for locally Lipschitz periodic functionals and related results. The key argument in our proofs is the Ekeland's Variational Principle and a non-smooth Pseudo-Gradient Lemma. As application of these abstract results we solve a non-linear setvalued elliptic problem.

Introduction

In PDE, two important tools for proving existence of solutions are the Mountain-Pass Theorem of Ambrosetti and Rabinowitz (and its various generalizations) and the Ljusternik-Schnirelmann Theorem. These results apply to the case when the solutions of the given problem are critical points of an appropriate functional of energy f, which is supposed to be real and C^1, or even differentiable, on a real Banach space X. One may ask what happens if f, which often is associated to the original equation in a canonical way, fails to be differentiable. In this case the gradient of f must be replaced by a generalized one, in a sense which is to be defined.

The first approach is due to Chang [8] and Aubin and Clarke [2], who considered the case of a locally Lipschitz function f. For such functions, Clarke [11] defined a generalized gradient, which coincides to the usual ones if f is differentiable or convex. Still denoting this generalized gradient by ∂f, critical points of f are all points x such that $0 \in \partial f(x)$. In this setting, Chang [8] proved a version of the Mountain Pass Lemma, in the case when X is reflexive. For this aim, he used a "Lipschitz version" of the Deformation Lemma. The same result was used for the proof of the Ljusternik-Schnirelmann Theorem in the Lipschitz case. As observed by Brézis, the reflexivity assumption on X is not necessary.

Our main result is a multiplicity theorem for locally Lipschitz periodic functionals, their set of periods being a discrete subgroup of the space where they are defined. This result can be regarded as a Ljusternik-Schnirelmann type theorem for non-differentiable functionals.

After recalling a well known theorem due to Choulli, Deville and Rhandi [9] and giving some consequences of this Mountain Pass type theorem for locally Lipschitz functionals, we present the connection between their theorem and our main result by solving a non-smooth problem that generalizes the forced-pendulum equation.

Following [8], authors usually impose measurability conditions to some *a priori* unknown functions in order to be able to find ∂f. We first show that these conditions are automatically fullfilled and then we prove the existence of critical points, which are shown to be solutions of a multivalued PDE.

1. The theoretical setting

Throughout, X will be a real Banach space. Let X^* be its dual and $\langle x^*, x\rangle$, for $x \in X$, $x \in X^*$, denote the duality pairing between X^* and X. We say that a function $f : X \to \mathbf{R}$ is locally Lipschitz ($f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$) if, for each $x \in X$, there is a neighbourhood V of x and a constant $k = k(V)$ depending on V such that

$$|f(y) - f(z)| \leq k\|y - z\| \quad ,$$

for each $y, z \in V$.

We recall in what follows the definition of the Clarke subdifferential and some of its most important properties (see [10] for details).

For each $x, v \in X$, we define the generalized directional derivative at x in the direction v of a given $f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$ as

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda} \quad .$$

Then $f^0(x, v)$ is a finite number and $|f^0(x, v)| \leq k\|v\|$. The mapping $v \longmapsto f^0(x, v)$ is positively homogeneous and subadditive, hence convex continuous. The generalized gradient (the Clarke subdifferential) of f at x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^*;\ \ f^0(x, v) \geq \langle x^*, v\rangle, \text{ for all } v \in X\} \quad .$$

If f is convex, $\partial f(x)$ coincides with the subdifferential of f at x in the sense of convex analysis.

The fundamental properties of the Clarke subdifferential are:

a) For each $x \in X$, $\partial f(x)$ is a nonempty convex weak-$\star$ compact subset of X^*.

b) For each $x, v \in X$, we have

$$f^0(x, v) = \max\{\langle x^*, v\rangle;\ x^* \in \partial f(x)\} \quad .$$

c) The set-valued mapping $x \longmapsto \partial f(x)$ is upper semi-continuous in the sense that for each $x_0 \in X, \varepsilon > 0, v \in X$, there is $\delta > 0$ such that for each $x^* \in \partial f(x)$ with $\|x - x_0\| < \delta$, there exists $x_0^* \in \partial f(x_0)$ such that $|\langle x^* - x_0^*, v \rangle| < \varepsilon$.

d) The function $f^0(\cdot, \cdot)$ is upper semi-continuous.

e) If f achieves a local minimum or maximum at x, then $0 \in \partial f(x)$.

f) The function

$$\lambda(x) = \min_{x^* \in \partial f(x)} \|x^*\|$$

exists and is lower semi-continuous.

g) Lebourg's Mean Value Theorem: If x and y are distinct points in X, then there is a point z in the open segment between x and y such that

$$f(y) - f(x) \in \langle \partial f(z), y - x \rangle \ .$$

Definition 1. *A point $u \in X$ is said to be a critical point of $f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$ if $0 \in \partial f(u)$, namely $f^0(u, v) \geq 0$ for every $v \in X$. A real number c is called a critical value of f if there is a critical point $u \in X$ such that $f(u) = c$.*

Definition 2. *If $f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$ and c is a real number, we say that f satisfies the Palais-Smale condition at the level c (in short $(PS)_c$) if any sequence (x_n) in X with the properties $\lim\limits_{n\to\infty} f(x_n) = c$ and $\lim\limits_{n\to\infty} \lambda(x_n) = 0$ has a convergent subsequence. The function f is said to satisfy the Palais-Smale condition (in short (PS)) if each sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lim\limits_{n\to\infty} \lambda(x_n) = 0$ has a convergent subsequence.*

Let Z be a discrete subgroup of X, that is

$$\inf_{z \in Z \setminus \{0\}} \|z\| > 0 \ .$$

A function $f : X \to \mathbf{R}$ is said to be *Z-periodic* if $f(x + z) = f(x)$, for every $x \in X$ and $z \in Z$.

If $f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$ is Z-periodic, then $x \longmapsto f^0(x, v)$ is Z-periodic, for all $v \in X$ and ∂f is Z-invariant, that is $\partial f(x + z) = \partial f(x)$, for every $x \in X$ and $z \in Z$. These implies that λ inherits the Z-periodicity property.

If $\pi : X \to X/Z$ is the canonical surjection and x is a critical point of f, then $\pi^{-1}(\pi(x))$ contains only critical points. Such a set is called a *critical orbit* of f. Note that X/Z is a complete metric space endowed with the metric

$$d(\pi(x), \pi(y)) = \inf_{z \in Z} \|x - y - z\| \ .$$

Definition 3. *A locally Lipschitz Z-periodic function $f : X \to \mathbf{R}$ is said to satisfy the $(PS)_Z$ - condition provided that, for each sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lambda(x_n) \to 0$, then $(\pi(x_n))$ is relatively compact in X/Z. If c is a real number, then f is* *said to satisfy the* $(PS)_{Z,c}$ *- condition if, for any sequence* (x_n) *in* X *such that* $f(x_n) \to c$ *and* $\lambda(x_n) \to 0$, *there is a convergent subsequence of* $(\pi(x_n))$.

Denote Cr (f, c) the set of critical points of the locally Lipschitz function $f : X \to \mathbf{R}$ at the level $c \in \mathbf{R}$, that is

$$\mathrm{Cr}\ (f, c) = \{x \in X;\ f(x) = c \ \text{ and } \ \lambda(x) = 0\} \ .$$

2. The main result

Theorem 1. *Let* $f : X \to \mathbf{R}$ *be a bounded below locally Lipschitz* Z*-periodic function with the* $(PS)_Z$*-property. Then* f *has at least* $n + 1$ *distinct critical orbits, where* n *is the dimension of the vector space generated by the discrete subgroup* Z.

Before beginning the proof, we shall recall the notion of *category* and some of its properties, which will be required by the proof of the main result.

A topological space X is said to be *contractible* if the identity of X is homotopical to a constant map, that is there exist $u_0 \in X$ and a continuous map $F : [0, 1] \times X \to X$ such that

$$F(0, \cdot) = \mathrm{id}_X \quad \text{and} \quad F(1, \cdot) = u_0 \ .$$

A subset M of X is said to be *contractible in* X if there exist $u_0 \in X$ and a continuous map $F : [0, 1] \times M \to X$ such that

$$F(0, \cdot) = \mathrm{id}_M \quad \text{and} \quad F(1, \cdot) = u_0 \ .$$

If A is a subset of X, we define the category of A in X as follows:

$\mathrm{Cat}_X(A) = 0, \quad$ if $\ A = \emptyset$.

$\mathrm{Cat}_X(A) = n, \quad$ if n is the smallest integer such that A can be covered by n closed sets which are contractible in X.

$\mathrm{Cat}_X(A) = \infty, \quad$ otherwise.

Lemma 1. *Let* A *and* B *subsets of* X. *Then the following hold:*

i) If $A \subset B$, *then* $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$.

ii) $\mathrm{Cat}_X(A \cup B) \le \mathrm{Cat}_X(A) + \mathrm{Cat}_X(B)$

iii) Let $h : [0, 1] \times A \to X$ *be a continuous mapping such that* $h(0, x) = x$ *for every* $x \in A$. *If* A *is closed and* $B = h(1, A)$, *then* $\mathrm{Cat}_X\ (A) \le \mathrm{Cat}_X\ (B)$

iv) If n *is the dimension of the vector space generated by the discrete group* Z, *then, for each* $1 \le i \le n + 1$, *the set*

$$\mathcal{A}_i = \{A \subset X;\ A \ \text{ is compact and } \ \mathrm{Cat}_{\pi(X)}\pi(A) \ge i\}$$

is nonempty. Obviously, $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots \supset \mathcal{A}_{n+1}$.

The only nontrivial part is *iv)* , which can be found in [19].

The following two Lemmas are proved in [26].

Lemma 2. *For each $1 \le j \le n+1$, the space $\mathcal{A}_i$ endowed with the Hausdorff metric*

$$\delta(A,B) = \max\{\sup_{a\in A} \operatorname{dist}(a,B)\ ,\ \sup_{b\in B} \operatorname{dist}(b,A)\}$$

is a complete metric space.

Lemma 3. *If $1 \le i \le n+1$ and $f \in C(X,\mathbf{R})$, then the function $\eta : \mathcal{A}_i \to \mathbf{R}$ defined by*

$$\eta(A) = \max_{x\in A} f(x)$$

is lower semi-continuous.

If n is the dimension of the vector space generated by the discrete group Z, one sets for each $1 \le i \le n+1$

$$c_i = \inf_{A\in\mathcal{A}_i} \eta(A) \quad .$$

For each $c \in \mathbf{R}$ we denote $[f \le c] = \{x \in X;\ f(x) \le c\}$.

3. Proof of Theorem 1

It follows from Lemma 1 *iv)* and the lower boundedness of f that

$$-\infty < c_1 \le c_2 \le \dots \le c_{n+1} < +\infty \quad .$$

It is sufficient to show that, if $1 \le i \le j \le n+1$ and $c_i = c_j = c$, then the set $\operatorname{Cr}(f,c)$ contains at least $j-i+1$ distinct critical orbits. We argue by contradiction and suppose that, for some $i \le j$, $\operatorname{Cr}(f,c)$ has $k \le j-i$ distinct critical orbits, generated by $x_1,\dots,x_k \in X$. We construct first an open neighbourhood of $\operatorname{Cr}(f,c)$ of the form

$$V_r = \bigcup_{l=1}^{k} \bigcup_{z\in Z} B(x_l+z,r) \quad .$$

Moreover, we may suppose that $r>0$ is chosen such that π is one-to-one on $\overline{B}(x_l,2r)$. This condition ensures that $\operatorname{Cat}_{\pi(X)}(\pi(\overline{B}(x_l,2r))) = 1$, for each $l = 1,\dots,k$. Here $V_r = \emptyset$ if $k=0$.

<u>*Step 1.*</u> We prove that there exists $0 < \varepsilon < \min\{\frac{1}{4}, r\}$ such that, for each $x \in [c-\varepsilon \le f \le c+\varepsilon] \setminus V_r$, one has

$$\lambda(x) > \sqrt{\varepsilon} \quad . \tag{1}$$

Indeed, if not, there is a sequence (x_m) in $X \setminus V_r$ such that, for each $m \geq 1$,

$$c - \frac{1}{m} \leq f(x_m) \leq c + \frac{1}{m} \quad \text{and} \quad \lambda(x_m) \leq \frac{1}{\sqrt{m}} \quad .$$

Since f satisfies $(PS)_Z$, it follows that, up to a subsequence, $\pi(x_m) \to \pi(x)$ as $m \to \infty$, for some $x \in X \setminus V_r$. By the Z-periodicity of f and λ, we can assume that $x_m \to x$ as $m \to \infty$. The continuity of f and the lower semi-continuity of λ imply $f(x) = c$ and $\lambda(x) = 0$, which is a contradiction, since $x \in X \setminus V_r$.

Step 2. For ε found above and according to the definition of c_j, there exists $A \in \mathcal{A}_j$ such that

$$\max_{x \in A} f(x) < c + \varepsilon^2 \quad .$$

Setting $B = A \setminus V_{2r}$, we get by Lemma 1 that

$$j \leq \mathrm{Cat}_{\pi(X)}(\pi(A)) \leq \mathrm{Cat}_{\pi(X)}(\pi(B) \cup \pi(\overline{V}_{2r})) \leq$$

$$\leq \mathrm{Cat}_{\pi(X)}(\pi(B)) + \mathrm{Cat}_{\pi(X)}(\pi(\overline{V}_{2r})) \leq \mathrm{Cat}_{\pi(X)}(\pi(B)) + k \leq \mathrm{Cat}_{\pi(X)}(\pi(B)) + j - i \ .$$

Hence, $\mathrm{Cat}_{\pi(X)}(\pi(B)) \geq i$, that is $B \in \mathcal{A}_i$.

Step 3. For ε and B as above we apply the Ekeland's Principle to the functional η defined in Lemma 3. It follows that there exists $C \in \mathcal{A}_i$ such that, for each $D \in \mathcal{A}_i$, $D \neq C$,

$$\eta(C) \leq \eta(B) \leq \eta(A) \leq c + \varepsilon^2 \quad ,$$

$$\delta(B, C) \leq \varepsilon \quad ,$$

$$\eta(D) > \eta(C) - \varepsilon \delta(C, D) \quad . \tag{2}$$

Since $B \cap V_{2r} = \emptyset$ and $\delta(B, C) \leq \varepsilon < r$, it follows that $C \cap V_r = \emptyset$. In particular, the set $F = [c - \varepsilon \leq f] \cap C$ is contained in $[c - \varepsilon \leq f \leq c + \varepsilon]$ and $F \cap V_r = \emptyset$.

Lemma 4. *Let M be a compact metric space and let $\varphi : M \to 2^{X^*}$ be a set-valued mapping which is upper semi-continuous (in the sense of c)) and with weak-$\star$ compact convex values. For $t \in M$ denote*

$$\gamma(t) = \inf\{\|x^*\|;\ x^* \in \varphi(t)\}$$

and

$$\gamma = \inf_{t \in M} \gamma(t) \quad .$$

Then, given $\varepsilon > 0$, there exists a continuous function $v : M \to X$ such that for all $t \in M$ and $x^* \in \varphi(t)$,

$$\|v(t)\| \leq 1 \text{ and } \langle x^*, v(t) \rangle \geq \gamma - \varepsilon \quad .$$

Proof of Lemma. We may suppose $\gamma > 0$ and $0 < \varepsilon < \gamma$. If B_r denotes the open ball in X^* centered at 0 with radius r, then, for each $t \in M$, one has

$$B_{\gamma-\frac{\varepsilon}{2}} \cap \varphi(t) = \emptyset \quad .$$

Since $\varphi(t)$ and $B_{\gamma-\frac{\varepsilon}{2}}$ are convex, weak-$\star$ compact and disjoint, it follows from the Theorem 3.4 in [24], applied to the space $(X^*,\ \sigma(X^*, X))$ and from the fact that the dual space of the above one is X, that:

for every $t \in M$, there is some $v_t \in X$, $\|v_t\| = 1$ such that

$$\langle \xi, v_t \rangle \leq \langle x^*, v_t \rangle \quad ,$$

for each $\xi \in B_{\gamma-\frac{\varepsilon}{2}}$ and $x^* \in \varphi(t)$. Therefore, for each $x^* \in \varphi(t)$,

$$\langle x^*, v_t \rangle \geq \sup_{\xi \in B_{\gamma-\frac{\varepsilon}{2}}} \langle \xi, v_t \rangle = \gamma - \frac{\varepsilon}{2} \quad .$$

Because of the upper semi-continuity of φ, there is an open neighbourhood $V(t)$ of t such that, for each $t' \in V(t)$ and each $x^* \in \varphi(t')$,

$$\langle x^*, v_t \rangle > \gamma - \varepsilon \quad .$$

Since M is compact and $M = \bigcup_{t \in M} V(t)$, we can find a finite subcovering $\{V_1, ..., V_n\}$ of M. Let $v_1, ..., v_n$ be on the unit sphere of X such that $\langle x^*, v_i \rangle > \gamma - \varepsilon$, for all $1 \leq i \leq n$, $t \in V_i$ and $x^* \in \varphi(t)$.

If $\rho_i(t) = \operatorname{dist}(t, \partial V_i)$, define

$$\zeta_i(t) = \frac{\rho_i(t)}{\sum_{j=1}^n \rho_j(t)} \quad \text{and} \quad v(t) = \sum_{i=1}^n \zeta_i(t) v_i \quad .$$

The function v is the desired mapping. □

Applying Lemma 4 to $\varphi = \partial f$ on F, we find a continuous map $v : F \to X$ such that, for all $x \in F$ and $x^* \in \partial f(x)$,

$$\|v(x)\| \leq 1 \quad \text{and} \quad \langle x^*, v(x) \rangle \geq \inf_{x \in F} \lambda(x) - \varepsilon \geq \inf_{x \in C} \lambda(x) - \varepsilon \geq \sqrt{\varepsilon} - \varepsilon \quad ,$$

where the last inequality is justified by (1).

It follows that, for each $x \in F$ and $x^* \in \partial f(x)$,

$$f^0(x, -v(x)) = \max_{x^* \in \partial f(x)} \langle x^*, -v(x) \rangle = -\min_{x^* \in \partial f(x)} \langle x^*, v(x) \rangle \leq \varepsilon - \sqrt{\varepsilon} < -\varepsilon,$$

from our choice of ε.

From the upper semi-continuity of f^0 and the compactness of F, there exists $\delta > 0$ such that if $x \in F$, $y \in X$, $\|y - x\| \leq \delta$, then

$$f^0(y, -v(x)) < -\varepsilon \quad . \tag{3}$$

Since $C \cap \mathrm{Cr}(f, c) = \emptyset$ and C is compact, while $\mathrm{Cr}(f, c)$ is closed, there is a continuous extension $w : X \to X$ of v such that $w|_{\mathrm{Cr}(f,c)} = 0$ and $\|w(x)\| \leq 1$, for all $x \in X$.

Let $\alpha : X \to [0, 1]$ be a continuous Z-periodic function such that $\alpha = 1$ on $[f \geq c]$ and $\alpha = 0$ on $[f \leq c - \varepsilon]$. Let $h : [0, 1] \times X \to X$ be the continuous mapping defined by

$$h(t, x) = x - t\delta\alpha(x)w(x) \quad .$$

If $D = h(1, C)$, it follows from Lemma 1 that

$$\mathrm{Cat}_{\pi(X)}(\pi(D)) \geq \mathrm{Cat}_{\pi(X)}(\pi(C)) \geq i$$

which shows that $D \in \mathcal{A}_i$, since D is compact.

Step 4. By Lebourg's mean value Theorem we get that, for each $x \in X$, there exists $\theta \in (0, 1)$ such that

$$f(h(1, x)) - f(h(0, x)) \in \langle \partial f(h(\theta, x)), -\delta\alpha(x)w(x)\rangle \quad .$$

Hence, there is some $x^* \in \partial f(h(\theta, x))$ such that

$$f(h(1, x)) - f(h(0, x)) = \alpha(x)\langle x^*, -\delta w(x)\rangle \quad .$$

It follows by (3) that, if $x \in F$, then

$$f(h(1, x)) - f(h(0, x)) = \delta\alpha(x)\langle x^*, -w(x)\rangle \leq \tag{4}$$

$$\leq \delta\alpha(x)f^0(x - \theta\delta\alpha(x)w(x), -v(x)\rangle \leq -\varepsilon\delta\alpha(x) \quad .$$

It follows that, for each $x \in C$,

$$f(h(1, x)) \leq f(x) \quad .$$

Let $x_0 \in C$ be such that $f(h(1, x_0)) = \eta(D)$. Hence,

$$c \leq f(h(1, x_0)) \leq f(x_0) \quad .$$

By the definition of α and F, it follows that $\alpha(x_0) = 1$ and $x_0 \in F$. Therefore, by (4), we get

$$f(h(1, x_0)) - f(x_0) \leq -\varepsilon\delta \quad .$$

Thus,

$$\eta(D) + \varepsilon\delta \leq f(x_0) \leq \eta(C) \quad . \tag{5}$$

Taking into account the definition of D, it follows that

$$\delta(C, D) \leq \delta \quad .$$

Therefore,

$$\eta(D) + \varepsilon\delta(C, D) \leq \eta(C) \quad ,$$

so that (2) implies $C = D$, which contradicts (5). □

4. A multivalued generalized version of the forced-pendulum problem

As an application of the above results, we shall study the periodic multivalued problem of the forced-pendulum

$$\begin{cases} x''(t) + f(t) \in [\underline{g}(x(t)), \overline{g}(x(t))] \ , \ \text{a.e. } t \in (0,1) \\ x(0) = x(1) \ , \end{cases} \tag{6}$$

where:

$$f \in L^p(0,1) \quad \text{for some} \ \ p > 1 \quad , \tag{7}$$

$$g \in L^\infty(\mathbf{R}), \ g(u+T) = g(u) \quad \text{for some} \ \ T > 0, \ \text{a.e.} \ \ u \in \mathbf{R} \quad , \tag{8}$$

$$\underline{g}(u) = \lim_{\varepsilon \searrow 0} \operatorname{essinf}\{g(u); \ |u - v| < \varepsilon\} \quad \overline{g}(u) = \lim_{\varepsilon \searrow 0} \operatorname{esssup}\{g(u); \ |u - v| < \varepsilon\} \quad ,$$

$$\int_0^T g(u)du = \int_0^1 f(t)dt = 0 \tag{9}$$

We shall prove

Theorem 2. *If f, g are as above, then the problem (6) has at least two solutions in*

$$X := H^1_p(0,1) = \{x \in H^1(0,1); \ \ x(0) = x(1)\} \quad ,$$

which are distinct in the sense that their difference is not an integer multiple of T.

Define the functional ψ in $L^\infty(0,1)$ by

$$\psi(x) = \int_0^1 (\int_0^{x(s)} g(u)du)ds \quad .$$

It is obvious that ψ is a Lipschitz map on $L^\infty(0,1)$.

Let $G(u) = \displaystyle\int_0^u g(v)dv$.

The following results show that the description of $\partial\psi$ given in [8] holds without further assumptions on g.

Lemma 5. *Let g be a locally bounded measurable function defined on* $\mathbf{R}$ *and* $\underline{g}$, $\overline{g}$ *as above. Then the Clarke subdifferential of* G *is given by*

$$\partial\, G(u) = [\underline{g}(u), \overline{g}(u)] \qquad u \in \mathbf{R} \ .$$

Proof. The required equality is equivalent to $G^0(u,1) = \overline{g}(u)$ and $G^0(u,-1) = \underline{g}(u)$.

As a matter of facts, examining the definitions of G^0, $\overline{g}$ and $\underline{g}$, it follows that $\underline{g}(u) = -(\overline{-g})(u)$ and $G^0(u,-1) = -(-G)^0(u,1)$, so that the second required equality is equivalent to the first one.

Now the inequality $G^0(u,1) \le \overline{g}(u)$ can be found in [8], so we have only to prove that $G^0(u,1) \ge \overline{g}(u)$. Suppose that $G^0(u,1) = \overline{g}(u) - \varepsilon$ for some $\varepsilon > 0$. Let $\delta > 0$ be such that

$$\frac{G(\tau+\lambda) - G(\tau)}{\lambda} < \overline{g}(u) - \frac{\varepsilon}{2} \ ,$$

if $|\tau - u| < \delta$ and $0 < \lambda < \delta$. Then

$$\frac{1}{\lambda}\int_{\tau}^{\tau+\lambda} g(s)ds < \overline{g}(u) - \frac{\varepsilon}{2} \qquad \text{if} \ \ |\tau - u| < \delta, \ \lambda > 0 \tag{10}$$

We claim that there exist $\lambda_n \searrow 0$ such that

$$\frac{1}{\lambda_n}\int_{\tau}^{\tau+\lambda_n} g(s)ds \to g(\tau) \qquad a.e. \ \ \tau \in (u-\delta, u+\delta) \ . \tag{11}$$

Suppose for the moment that (11) has already been proved. Now (10) and (11) show that

$$g(\tau) \le \overline{g}(u) - \frac{\varepsilon}{2} \qquad \text{if} \ \ \tau \in (u-\delta, u+\delta) \ \ ,$$

so we obtain the contradictory inequalities

$$\overline{g}(u) \le \operatorname{esssup}\{g(s); \ s \in [u-\delta, u+\delta]\} \le \overline{g}(u) - \frac{\varepsilon}{2} \ \ .$$

All it remains to be proved is (11). Note that we may cut g in order to suppose that $g \in L^\infty \cap L^1$. Then (11) is nothing that the classical fact that for each $\varphi \in L^1(\mathbf{R})$,

$$T_\lambda(\varphi) \to \varphi \qquad \text{as} \ \ \lambda \searrow 0 \tag{12}$$

where

$$T_\lambda\varphi(u) = \frac{1}{\lambda}\int_u^{u+\lambda} \varphi(s)ds \qquad \text{for} \ \ \lambda > 0, \ u \in \mathbf{R}, \ \varphi \in L^1(\mathbf{R}) \ \ .$$

Indeed, it can be easily seen that T_λ is linear and continuous in $L^1(\mathbf{R})$ and $\lim\limits_{\lambda \searrow 0} T_\lambda \varphi = \varphi$ in $\mathcal{D}(\mathbf{R})$ for $\varphi \in \mathcal{D}(\mathbf{R})$. Now (14) follows by a density argument. $\square$

Returning to our problem, it follows by Theorem 2.1. in [8] that

$$\partial\psi_{|H_0^1(\Omega)}(x) \subset \partial\psi(x) \tag{13}$$

In order to obtain information on $\partial\psi$, we shall need an improvement of the Theorem 2.1. in [8].

Theorem 3. *If $x \in L^\infty(0,1)$, then*

$$\partial\psi(x)(t) \subset [\underline{g}(x(t)), \overline{g}(x(t))] \quad a.e. \;\; t \in (0,1) \;\; ,$$

in the sense that if $w \in \partial\psi(x)$ then

$$\underline{g}(x(t)) \le w(t) \le \overline{g}(x(t)) \qquad a.e. \;\; t \in (0,1) \tag{14}$$

Proof. Let h be a Borel function such that $h = g$ a.e. on $\mathbf{R}$. It follows that the set

$$A = \{t \in (0,1) \; ; \;\; \underline{g}(x(t)) \ne \underline{h}(x(t))\}$$

is a null set. (A similar reasoning can be done for $\overline{g}$ and $\overline{h}$).

Therefore we may suppose that g is a Borel function. We would like to deal with $\displaystyle\int_0^1 \overline{g}(x(t))dt$, so we have to prove that $\overline{g}$ is a Borel function.

Lemma 6. *Let $g : \mathbf{R} \to \mathbf{R}$ be a locally bounded Borel function. Then $\overline{g}$ is a Borel function.*

Proof of Lemma. Since the requirement is local, we may suppose that g is bounded by 1, for example, and it is nonnegative. Since

$$g = \lim_{n\to\infty} \lim_{m\to\infty} g_{m,n}$$

where

$$g_{m,n}(x,t) = \Big(\int_{t-\frac{1}{n}}^{t+\frac{1}{n}} |g^m(x,s)| ds \Big)^{\frac{1}{m}}$$

it suffices to prove that $g_{m,n}$ is Borel.

Let

$$\mathcal{M} = \{g : \Omega \times \mathbf{R} \to \mathbf{R}; \;\; |g| \le 1 \;\text{ and }\; g \;\text{ is a Borel function}\}$$

$$\mathcal{N} = \{g \in \mathcal{M}; \;\; g_{m,n} \;\text{ is a Borel function}\}$$

It is known (see [3], p. 178) that $\mathcal{M}$ is the smallest set of functions having the following properties:

i) $\{g \in C(\Omega \times \mathbf{R}, \mathbf{R}); \quad |g| \leq 1\} \subset \mathcal{M}$

ii) $g^{(k)} \xrightarrow{k} g$ imply $g \in \mathcal{M}$. Note that here we have an "each point" convergence.

Since $\mathcal{N}$ contains obviously the continuous functions and ii) is also true for $\mathcal{N}$, by the Dominated Convergence Theorem, it follows that $\mathcal{M} = \mathcal{N}$. □

Proof of Theorem 3 continued. Let $v \in L^\infty(\Omega),\ v \geq 0$. Then, for suitable $\lambda_i \searrow 0$ and $h_i \to 0$ in $L^{p+1}(\Omega)$ one has

$$\psi^0(u,v) = \lim_{i\to\infty} \frac{1}{\lambda_i} \int_\Omega \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)ds\ dx$$

We may suppose that $h_i \to 0$ a.e., so that

$$\psi^0(u,v) = \lim_{i\to\infty} \frac{1}{\lambda_i} \int_{[v>0]} \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)ds\ dx \leq$$

$$\leq \int_{[v>0]} (\limsup_{i\to\infty} \frac{1}{\{\lambda_i} \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)ds)dx \leq$$

$$\leq \int_{[v>0]} \overline{g}(x,u(x))v(x)dx$$

so that

$$\psi^0(u,v) \leq \int_{[v>0]} \overline{g}(x,u(x))v(x)dx \tag{15}$$

for such v.

Suppose now that (21) is false, that is, for example, there exist $\varepsilon > 0$, a set E with $|E| > 0$ and $w \in \partial\psi(u)$ such that

$$w(x) \geq \overline{g}(x,u(x)) + \varepsilon \qquad \text{on } E \tag{16}$$

Now (15) with $v = \mathbf{1}_E$ shows that

$$\langle w, v\rangle = \int_E w \leq \psi^0(u,v) \leq \int_E \overline{g}(x,u(x))dx$$

which contradicts (16). □

Proof of Theorem 2. Define on the space $X = H^1_{per}(0,1)$ the locally Lipschitz function

$$\varphi(x) = \frac{1}{2}\int_0^1 x'^2(t)dt - \int_0^1 f(t)x(t)dt + \int_0^1 G(x(t))dt.$$

The critical points of φ are solutions of (6). Indeed, it is obvious that

$$\partial\varphi(x) = x'' + f - \partial\psi_{|H^1_{per}(0,1)}(x) \quad \text{in} \quad H^{-1}(0,1)$$

If x_0 is a critical point of φ it follows that there exists $w \in \partial\psi_| H^1_{per}(0,1)(x_0)$ such that

$$x'' + f = w \quad \text{in} \quad H^{-1}(0,1)$$

Since $\varphi(x+T) = \varphi(x)$, we are going to use the Theorem 1. All we have to do is to verify the $(PS)_{Z,c}$ condition, for each c, and to prove that (6) has a solution x_0 that minimizes φ on $H^1_{per}(0,1)$. Note first that every $x \in H^1_{per}(0,1)$ can be written

$$x(t) = \int_0^1 x(s)ds + \overline{x}(t)dt \quad \text{with } \overline{x} \in H^1_0(0,1).$$

Hence, by the Poincaré's inequality,

$$\varphi(x) = \frac{1}{2}\int_0^1 \overline{x}'^2(t)dt - \int_0^1 f(t)\overline{x}(t)dt + \int_0^1 G(x(t))dt$$

$$\geq \frac{1}{2}\|\overline{x}'^2\|^2_{L^2} - \|f\|_{L^p}\cdot\|\overline{x}\|_{L^{p'}} - \|G\|_{L^\infty}$$

$$\geq \frac{1}{2}\|\overline{x}'^2\|^2_{L^2} - C\,\|f\|_{L^p}\cdot\|\overline{x}'\|_{L^2} - \|G\|_{L^\infty} \to \infty \quad \text{as } \|\overline{x}\|_{H^1} \to \infty\,,$$

where p' denotes the conjugated exponent of p.

We verify in what follows the $(PS)_{Z,c}$ condition, for each c. Let $(x_n) \subset X$ be such that

$$\varphi(x_n) \to c \tag{17}$$

$$\lambda(x_n) \to 0. \tag{18}$$

Let $w_n \in \partial\varphi(x_n) \subset L^\infty(0,1)$ (because $\underline{g}\circ x_n \leq w_n \leq \overline{g}\circ x_n$ and $\underline{g},\overline{g} \in L^\infty(\mathbf{R})$) be such that

$$\lambda(x_n) = x_n'' + f - w_n \to 0 \quad \text{in} \quad H^{-1}(0,1)$$

Then, multiplying (18) by x_n we get

$$\int_0^1 (x_n')^2 - \int_0^1 fx_n + \int_0^1 w_n x_n = o(1)\|x_n\|_{H^1_p}$$

and, by (17),

$$-\frac{1}{2}\int_0^1 (x_n')^2 + \int_0^1 fx_n - \int_0^1 G(x_n), \to c$$

so that there exist positive constants C_1, C_2 such that

$$\int_0^1 (x_n')^2 \leq C_1 + C_2 \|x_n\|_{H_p^1}$$

Note that G is also T-periodic; hence it is bounded.

Replacing x_n by $x_n + kT$ for a suitable integer k, we may suppose that

$$x_n(0) \in [0, T]$$

so that (x_n) is bounded in H_p^1.

Let $x \in H_p^1$ be such that, up to a subsequence, $x_n \rightharpoonup x$ and $x_n(0) \to x(0)$. Then

$$\int_0^1 (x_n')^2 = \langle -x_n'' - f + w_n, x_n - x \rangle + \int_0^1 w_n(x_n - x) -$$

$$- \int_0^1 f(x_n - x) + \int_0^1 x_n' x' \to \int_0^1 x'^2$$

because $x_n \to x$ in $L^{p'}$, where p' is the conjugated exponent of p. It follows that $x_n \to x$ in H_p^1. □

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Nontrivial solutions for a multivalued problem with strong resonance

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The Mountain-Pass Theorem of Ambrosetti and Rabinowitz ([1]) and the Saddle Point Theorem of Rabinowitz ([21]) are very important tools in the critical point theory of C^1-functionals. That is why it is natural to ask us what happens if the functional fails to be differentiable. The first who considered such a case were Aubin and Clarke ([6]) and Chang ([12]), who gave suitable variants of the Mountain-Pass Theorem for locally Lipschitz functionals which are defined on reflexive Banach spaces. For this aim they replaced the usual gradient with a generalized one, which was firstly defined by Clarke ([13], [14]). As observed by Brezis ([12], p. 114), these abstract critical point theorems remain valid in non-reflexive Banach spaces.

We apply some of these results to solve a multivalued problem with strong resonance at infinity. We remark that it is not natural to consider nonlinearities which are strongly resonant at $+\infty$, but which may not be strongly resonant at $-\infty$. The literature is very rich in resonant problems, the first who studied such problems, however in the smooth case, being Landesman and Lazer ([18]). They found sufficient conditions for the existence of solutions for some singlevalued equations with Dirichlet conditions. These problems, that arise frequently in mechanics, were thereafter intensively studied and many applications to concrete situations were given.

1 Abstract framework

Let X be a real Banach space and let $f : X \to \mathbf{R}$ be a locally Lipschitz function. For each $x, v \in X$, we define the *generalized directional derivative* of f at x in the direction v as

$$f^0(x,v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y+\lambda v) - f(y)}{\lambda} .$$

The generalized gradient (the Clarke subdifferential) of f at the point x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^*;\ f^0(x,v) \geq \langle x^*, v\rangle,\ \text{for all } v \in X\} .$$

We also define the lower semi-continuous function

$$\lambda(x) = \min\{\|x^*\|;\ x^* \in \partial f(x)\} .$$

For further properties of these notions we refer to [12, 13, 14].

We say that a point $x \in X$ is a critical point of f provided that $0 \in \partial f(x)$, that is $f^0(x,v) \geq 0$ for every $v \in X$. If c is a real number, we say that f satisfies the Palais-Smale condition at the level c (in short $(PS)_c$) if any sequence $(x_n)_n$ in X with the properties $\lim_{n\to\infty} f(x_n) = c$ and $\lim_{n\to\infty} \lambda(x_n) = 0$ is relatively compact.

We shall use in this paper the following result, which is an imediate consequence of the Mountain Pass Theorem proved in [12].

THEOREM 1. *Let $f : X \to \mathbf{R}$ be a locally Lipschitzian function. Suppose that $f(0) = 0$ and there is some $v \in X \setminus \{0\}$ such that $f(v) \leq 0$. Moreover, assume that f satisfies the following geometric hypothesis: there exist $0 < R < \|v\|$ and $\alpha > 0$ such that, for each $u \in X$ with $\|u\| = R$, we have $f(u) \geq \alpha$.*

Let $\mathcal{P}$ be the family of all continuous paths $p : [0,1] \to X$ that join 0 to v and

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} f(p(t)) \ .$$

Then there exists a sequence (x_n) in X such that:

(i) $\lim_{n\to\infty} f(x_n) = c$;

(ii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

Moreover, if f satisfies $(PS)_c$ then c is a critical value of f.

The following Saddle Point type result generalizes the Rabinowitz's Theorem ([21]). Its proof is an easy exercise and is left to the reader.

THEOREM 2. *Let $f : X \to \mathbf{R}$ be a locally Lipschitzian function. Assume that $X = Y \oplus Z$, where Z is a finite dimensional subspace of X and for some $z_0 \in Z$ there exists $R > \|z_0\|$ such that*

$$\inf_{y\in Y} f(y + z_0) > \max\{f(z);\ z \in Z,\ \|z\| = R\} \ ,$$

Let

$$K = \{z \in Z;\ \|z\| \leq R\}$$

and

$$\mathcal{P} = \{p \in C(K,X);\ p(x) = x \text{ if } \|x\| = R\} \ .$$

If c is defined as in Theorem 1 and f satisfies $(PS)_c$, then c is a critical value of f.

2 Main results

Let M be a m-dimensional smooth compact Riemann manifold, possibly with smooth boundary ∂M. Particularly, M can be any open bounded smooth subset of $\mathbf{R}^m$. We shall consider the following multivalued elliptic problem

$$(P) \qquad \begin{cases} -\Delta_M u(x) - \lambda_1 u(x) \in [\,\underline{f}(u(x)), \overline{f}(u(x))\,] & \text{a.e. } x \in M \\ u = 0 & \text{on } \partial M \\ u \not\equiv 0 & \end{cases}$$

where:

i) Δ_M is the Laplace-Beltrami operator on M.

ii) λ_1 is the first eigenvalue of $-\Delta_M$ in $H_0^1(M)$.

iii) $f \in L^\infty(\mathbf{R})$.

iv) $\underline{f}(t) = \lim\limits_{\varepsilon \searrow 0} essinf\,\{f(s);\ |t-s| < \varepsilon\}$

$\overline{f}(t) = \lim\limits_{\varepsilon \searrow 0} esssup\,\{f(s);\ |t-s| < \varepsilon\}$.

As proved in [12], the functions $\underline{f}$ and $\overline{f}$ are measurable on $\mathbf{R}$ and, if

$$F(t) = \int_0^t f(s)ds ,$$

then the Clarke subdifferential of F is given by

$$\partial F(t) = [\,\underline{f}(t), \overline{f}(t)\,] \quad \text{a.e. } t \in \mathbf{R} .$$

Let $(g_{ij}(x))_{i,j}$ define the metric on M. We consider on $H_0^1(M)$ the locally Lipschitz functional $\varphi = \varphi_1 - \varphi_2$, where

$$\varphi_1(u) = \frac{1}{2}\int_M (\sum_{i,j} g_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \lambda_1 u^2\ )dx \quad \text{and} \quad \varphi_2(u) = \int_M F(u)dx .$$

By a solution of the problem (P) we shall mean any critical point of the energetic functional φ.

Denote

$$f(\pm\infty) = ess \lim_{t\to\pm\infty} f(t) \qquad \text{and} \qquad F(\pm\infty) = \lim_{t\to\pm\infty} F(t) .$$

Our basic hypothesis on f will be

$$(f1) \qquad f(+\infty) = F(+\infty) = 0 ,$$

which makes the problem (P) a Landesman-Lazer type one, with strong resonance at $+\infty$.

The following formulates a sufficient condition for the existence of solutions of our problem:

THEOREM A. *Assume that f satisfies (f1) and either*

$$(F1) \qquad F(-\infty) = -\infty$$

or $-\infty < F(-\infty) \le 0$ and there exists $\eta > 0$ such that

$$(F2) \qquad F \quad \textit{is non-negative on } (0,\eta) \textit{ or } (-\eta, 0)$$

Then the problem (P) has at least one solution.

For positive values of $F(-\infty)$ it is necessary to impose additional restrictions to f. Our variant for this case is

THEOREM B. *Assume (f1) and $0 < F(-\infty) < +\infty$.*

Then the problem (P) has at least one solution provided the following conditions are satisfied:

$$f(-\infty) = 0$$

and

$$F(t) \leq \frac{\lambda_2 - \lambda_1}{2} t^2 \qquad \textit{for each} \quad t \in \mathbf{R} .$$

For the proof of Theorem A we shall make use of the following non-smooth variants of Lemmas 6 and 7 in [15] (see also [3] for Lemma 1) which can be obtained in the same manner:

LEMMA 1. *Assume $f \in L^\infty(\mathbf{R})$ and there exist $F(\pm\infty) \in \overline{\mathbf{R}}$. Moreover, suppose that*
(i) $f(+\infty) = 0$ if $F(+\infty)$ is finite;
and
(ii) $f(-\infty) = 0$ if $F(-\infty)$ is finite.
Then

$$\mathbf{R} \setminus \{a \cdot \textit{meas}\,(M);\ a = -F(\pm\infty)\} \subset \{c \in \mathbf{R};\ \varphi \quad \textit{satisfies} \quad (PS)_c\}$$

LEMMA 2. *Assume f satisfies (f1). Then φ satisfies $(PS)_c$, whenever $c \neq 0$ and $c < -F(-\infty) \cdot meas(M)$.*

Here meas$\,(M)$ denotes the Riemannian measure of M.

Proof of Theorem A. We shall develop some of the ideas used in [26]. There are two distinct situations:

Case 1. $F(-\infty)$ is finite, that is $-\infty < F(-\infty) \leq 0$. In this case, φ is bounded from below since

$$\varphi(u) = \frac{1}{2}\int_M \Big(\sum_{i,j} g_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \lambda_1 u^2\Big)dx - \int_M F(u)dx$$

and, by our hypothesis on $F(-\infty)$,

$$\sup_{u \in H_0^1(M)} \int_M F(u)dx < +\infty .$$

Therefore,

$$-\infty < a := \inf_{u \in H_0^1(M)} \varphi(u) \leq 0 = \varphi(0) .$$

Choose c small enough in order to have $F(ce_1) < 0$ (note that c may be taken positive if $F > 0$ in $(0, \eta)$ and negative if $F < 0$ in $(-\eta, 0)$). Here $e_1 > 0$ denotes the first eigenfunction of $-\Delta_M$ in $H_0^1(M)$. Hence $\varphi(ce_1) < 0$, so $a < 0$. It follows now from Lemma 2 that φ satisfies $(PS)_a$. The proof ends in this case by applying Theorem 1.

Case 2. $F(-\infty) = -\infty$. Then, by Lemma 1, φ satisfies $(PS)_c$ for each $c \neq 0$.

Let V be the orthogonal complement of the space spanned by e_1 with respect to $H_0^1(M)$, that is

$$H_0^1(M) = Sp\ \{e_1\} \oplus V .$$

For fixed $t_0 > 0$, denote

$$V_0 = \{t_0 e_1 + v;\ v \in V\} \qquad \text{and} \qquad a_0 = \inf_{v \in V_0} \varphi(v)\ .$$

Note that φ is coercive on V. Indeed, if $v \in V$, then

$$\varphi(v) \geq \frac{1}{2}(1 - \frac{\lambda_1}{\lambda_2})\|v\|^2_{H^1_0} - \int_M F(v) \to +\infty \quad \text{as } \|v\|_{H^1_0} \to +\infty,$$

because the first term has a quadratic growth at infinity (t_0 being fixed), while $\int_M F(v)$ is uniformly bounded (in v), in view of the behaviour of F near $\pm\infty$. Thus, a_0 is attained, because of the coercivity of φ on V. From the boundedness of φ on $H^1_0(M)$ it follows that $-\infty < a \leq 0 = \varphi(0)$ and $a \leq a_0$.

Again, there are two posibilities:

(i) $a < 0$. In this case, by Lemma 2, φ satisfies $(\mathrm{PS})_a$. Hence $a < 0$ is a critical value of φ.

(ii) $a = 0 \leq a_0$. Then, either $a_0 = 0$ or $a_0 > 0$. In the first case, as we have already remarked, a_0 is attained. Thus, there is some $v \in V$ such that

$$0 = a_0 = \varphi(t_0 e_1 + v)\ .$$

Hence, $u = t_0 e_1 + v \in H^1_0(M) \setminus \{0\}$ is a critical point of φ, that is a solution of (P).

If $a_0 > 0$, notice that φ satisfies $(\mathrm{PS})_b$ for each $b \neq 0$. Since $\lim_{t \to +\infty} \varphi(te_1) = 0$, we may apply Theorem 2 to conclude that φ has a critical value $c \geq a_0 > 0$. ∎

Proof of Theorem B. If V has the same signification as above, let

$$V_+ = \{te_1 + v;\ t > 0,\ v \in V\}\ .$$

It will be sufficient to show that the functional φ has a non-zero critical point. To do this, we shall make use of two different arguments.

If $u = te_1 + v \in V_+$ then

$$\varphi(u) = \frac{1}{2}\int_M (\,|\nabla v|^2 - \lambda_1 v^2\,) - \int_M F(te_1 + v)\ .$$

In view of the boundedness of F it follows that

$$-\infty < a_+ := \inf_{u \in V_+} \varphi(u) \leq 0\ .$$

We analyse two distinct situations:

Case 1. $\quad a_+ = 0$.

To prove that φ has a critical point, we use the same arguments as in the proof of Theorem A (the second case). More precisely, for some fixed $t_0 > 0$ we define at the same way V_0 and a_0. Obviously, $a_0 \geq 0 = a_+$, since $V_0 \subset V_+$. The proof follows from now on the same ideas as in Case 2 of Theorem A, by considering the two distinct situations $a_0 > 0$ and $a_0 = 0$.

Case 2. $a_+ < 0$.

Let $u_n = t_n e_1 + v_n$ be a minimizing sequence of φ in V_+. We observe that the sequences $(u_n)_n$ and $(v_n)_n$ are bounded. Indeed, this is essentially a compactness condition and may be proved in a similar way to Lemma 1. It follows that there exists $w \in \overline{V}_+$, such that, going eventually to a subsequence,

$$u_n \rightharpoonup w \quad \text{weakly in } H_0^1(M) \ .$$

$$u_n \to w \quad \text{strongly in } L^2(M) \ .$$

$$u_n \to w \quad \text{a.e.}$$

Applying the Lebesgue Dominated Convergence Theorem we obtain

$$\lim_{n\to\infty} \varphi_2(u_n) = \varphi_2(w) \ .$$

On the other hand,

$$\varphi(w) \le \liminf_{n\to\infty} \varphi_1(u_n) - \lim_{n\to\infty} \varphi_2(u_n) = \liminf_{n\to\infty} \varphi(u_n) = a_+ \ .$$

It follows that, necessarily, $\varphi(w) = a_+ < 0$. Since the boundary of V_+ is V and

$$\inf_{u\in V} \varphi(u) = 0 \ ,$$

we conclude that w is a local minimum of φ on V_+ and $w \in V_+$. ∎

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A nonsmooth critical point theory approach to some nonlinear elliptic equations in $\mathbf{R}^n$

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Abstract

We determine nontrivial solutions of some semilinear and quasilinear elliptic problems on $\mathbf{R}^n$; we make use of two different nonsmooth critical point theories which allow to treat two kinds of nonlinear problems. A comparison between the possible applications of the two theories is also made.

1 Introduction

Consider a functional J defined on some Banach space B and having a mountain-pass geometry: the celebrated theorem by Ambrosetti-Rabinowitz [1] states that if $J \in C^1(B)$ and J satisfies the Palais-Smale condition (PS condition in the sequel) then J admits a nontrivial critical point. In this paper we drop these two assumptions: in order to determine nontrivial solutions of some nonlinear elliptic equations in $\mathbf{R}^n$ ($n \geq 3$), we use the mountain-pass principle for a class of nonsmooth functionals which do not satisfy the PS condition. More precisely, we consider a model elliptic problem first studied by Rabinowitz [14] with the C^1-theory and we extend his results by means of the nonsmooth critical point theories of Clarke [6, 7] and Degiovanni et al. [9, 10]: one of the purposes of this paper is to emphasize some differences between these two theories. This study was inspired by previous work on the existence of standing wave solutions of nonlinear Schrödinger equations: after making a standing wave ansatz, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation

$$-\Delta u + b(x)u = f(x,u) \qquad \text{in } \mathbf{R}^n \tag{1}$$

under suitable conditions on b and assuming that f is smooth, superlinear and subcritical.

To explain our results we introduce some functional spaces. We denote by L^p the space of measurable functions u of p-th power absolutely summable on $\mathbf{R}^n$, that is, satisfying $\|u\|_p^p := \int_{\mathbf{R}^n} |u|^p < +\infty$; by H^1 we denote the Sobolev space normed by $\|u\|_{H^1}^2 := \int_{\mathbf{R}^n}(|Du|^2 + |u|^2)$. We will assume that the function b in (1) is greater than some positive constant; then we define the Hilbert space E of all functions $u : \mathbf{R}^n \to \mathbf{R}$ with $\|u\|_E^2 := \int_{\mathbf{R}^n} (|Du|^2 + b(x)u^2) < \infty$. We denote by E^* the dual space of E: as E is continuously embedded in H^1 we also have $H^{-1} \subset E^*$.

We first consider the case where $(-\Delta)$ in (1) is replaced by a quasilinear elliptic operator: we seek positive weak solutions $u \in E$ of the problem

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u)D_i u D_j u + b(x)u = f(x,u) \qquad \text{in } \mathbf{R}^n . \tag{2}$$

Note that if $a_{ij}(x,s) \equiv \delta_{ij}$, then (2) reduces to (1). Here and in the sequel, by positive solution we mean a nonnegative nontrivial solution. To determine weak solutions of (2) we look for critical points of the functional $J : E \to \mathbf{R}$ defined by

$$J(u) = \frac{1}{2}\int_{\mathbf{R}^n} \sum_{i,j=1}^{n} a_{ij}(x,u)D_i u D_j u \; + \; \frac{1}{2}\int_{\mathbf{R}^n} b(x)u^2 \; - \; \int_{\mathbf{R}^n} F(x,u) \qquad \forall u \in E\,,$$

where $F(x,s) = \int_0^s f(x,t)dt$. Under reasonable assumptions on a_{ij}, b, f, the functional J is continuous but not even locally Lipschitz, see [4]: therefore, we cannot work in the classical framework of critical point theory. Nevertheless, the Gâteaux-derivative of J exists in the smooth directions, i.e. for all $u \in E$ and $\varphi \in C_c^\infty$ we can define

$$J'(u)[\varphi] = \int_{\mathbf{R}^n} \left(\sum_{i,j=1}^{n} \left[a_{ij}(x,u)D_i u D_j \varphi + \frac{1}{2}\frac{\partial a_{ij}}{\partial s}(x,u)D_i u D_j u \varphi \right] + b(x)u\varphi - f(x,u)\varphi \right) .$$

According to the nonsmooth critical point theory developed in [9, 10] we know that critical points u of J satisfy $J'(u)[\varphi] = 0$ for all $\varphi \in C_c^\infty$ and hence solve (2) in distributional sense; moreover, since

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + b(x)u - f(x,u) \in E^*$$

we also have

$$\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u)D_i u D_j u \in E^*$$

and (2) is solved in the weak sense ($\forall \varphi \in E$). We refer to [4] for the adaptation of this theory to quasilinear equations of the kind of (2) and to [8, 11] for applications in the case of unbounded domains and for further references. Under suitable assumptions on a_{ij}, b, f and by using the above mentioned tools we will prove that (2) admits a positive weak solution.

Next, we take into account the case where f is not continuous: let $f(x,\cdot) \in L^\infty_{\text{loc}}(\mathbf{R})$ and denote

$$\underline{f}(x,s) = \lim_{\varepsilon \searrow 0} \text{ essinf } \{f(x,t);\ |t-s| < \varepsilon\} \qquad \overline{f}(x,s) = \lim_{\varepsilon \searrow 0} \text{ esssup } \{f(x,t);\ |t-s| < \varepsilon\}\ ;$$

our aim is to determine $u \in E$ such that

$$-\Delta u + b(x)u \in [\underline{f}(x,u), \overline{f}(x,u)] \qquad \text{in } \mathbf{R}^n . \tag{3}$$

Positive solutions u of (3) satisfy

$$0 \in \partial I(u) \ ,$$

where

$$I(u) \ = \ \frac{1}{2}\int_{\mathbf{R}^n}(|Du|^2 + b(x)u^2) \ - \ \int_{\mathbf{R}^n} F(x,u^+) \qquad \forall u \in E$$

and $\partial I(u)$ stands for the Clarke gradient [6, 7] of the locally Lipschitz energy functional I; more precisely,

$$\partial I(u) = \left\{ \zeta \in E^*; \ I^0(u;v) \geq \langle \zeta, v \rangle \ , \quad \forall v \in E \right\} \ ,$$

where

$$I^0(u;v) = \limsup_{\substack{w \to u \\ \lambda \searrow 0}} \frac{I(w+\lambda v) - I(w)}{\lambda} \ . \tag{4}$$

This problem may be reformulated, equivalently, in terms of hemivariational inequalities as follows: find $u \in E$ such that

$$\int_{\mathbf{R}^n}(DuDv + b(x)uv) + \int_{\mathbf{R}^n}(-F)^0(x,u;v) \geq 0 \qquad \forall v \in E \ , \tag{5}$$

where $(-F)^0(x,u;v)$ denotes the Clarke directional derivative of $(-F)$ at $u(x)$ with respect to $v(x)$ and is defined as in (4). So, when $f(x,\cdot)$ is not continuous, Clarke's theory will enable us to prove that (3) admits a positive solution.

The two existence results stated in next section have several points in common: in both cases we first prove that the corresponding functional has a mountain-pass geometry and that a PS sequence can be built at a suitable infmax level. Then we prove that the PS sequence is bounded and that its weak limit is a solution of the problem considered; the final step is to prove that this solution is not the trivial one: to this end we use the concentration-compactness principle [12] and the behaviour of the function b at infinity. However, the construction of a PS sequence and the proof that its weak limit is a solution are definitely different: they highlight the different tools existing in the two theories.

2 Main results

Let us first state our results concerning (2). We require the coefficients a_{ij} $(i,j = 1,...,n)$ to satisfy

$$\begin{cases} a_{ij} \equiv a_{ji} \\ a_{ij}(x,\cdot) \in C^1(\mathbf{R}) \quad \text{for a.e. } x \in \mathbf{R}^n \\ a_{ij}(x,s), \ \frac{\partial a_{ij}}{\partial s}(x,s) \in L^\infty(\mathbf{R}^n \times \mathbf{R}) \ ; \end{cases} \tag{6}$$

moreover, on the matrices $[a_{ij}(x,s)]$ and $[s\frac{\partial a_{ij}}{\partial s}(x,s)]$ we make the following assumptions:

$$\exists \nu > 0 \quad \text{such that} \quad \sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{for a.e. } x \in \mathbf{R}^n, \ \forall s \in \mathbf{R}, \ \forall \xi \in \mathbf{R}^n \tag{7}$$

$$\begin{cases} \exists \mu \in (2, 2^*) \ , \quad \gamma \in (0, \mu - 2) \quad \text{such that} \\ 0 \le s \displaystyle\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \le \gamma \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \quad \text{for a.e. } x \in \mathbf{R}^n, \ \forall s \in \mathbf{R}, \ \forall \xi \in \mathbf{R}^n \ . \end{cases} \tag{8}$$

We require that $b \in L^\infty_{\text{loc}}(\mathbf{R}^n)$ and that

$$\begin{cases} \exists \underline{b} > 0 \quad \text{such that} \quad b(x) \ge \underline{b} \quad \text{for a.e. } x \in \mathbf{R}^n \\ \text{ess} \displaystyle\lim_{|x| \to \infty} b(x) = +\infty \ . \end{cases} \tag{9}$$

Let μ be as in (8), assume that $f(x,s) \not\equiv 0$ and

$$\begin{cases} f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R} \text{ is a Carathéodory function} \\ f(x,0) = 0 \qquad \text{for a.e. } x \in \mathbf{R}^n \\ 0 \le \mu F(x,s) \le s f(x,s) \qquad \forall s \ge 0 \text{ and for a.e. } x \in \mathbf{R}^n \ ; \end{cases} \tag{10}$$

moreover, we require f to be subcritical

$$\begin{cases} \forall \varepsilon > 0 \quad \exists f_\varepsilon \in L^{\frac{2n}{n+2}}(\mathbf{R}^n) \text{ such that} \\ |f(x,s)| \le f_\varepsilon(x) + \varepsilon |s|^{\frac{n+2}{n-2}} \qquad \forall s \in \mathbf{R} \text{ and for a.e. } x \in \mathbf{R}^n \ . \end{cases} \tag{11}$$

Finally, for all $\delta \in (2, 2^*)$ define $q(\delta) = \frac{2n}{2n+(2-n)\delta}$: then we assume[1]

$$\begin{cases} \exists C \ge 0 \ , \quad \exists \delta \in (2, 2^*) \ , \quad \exists G \in L^{q(\delta)}(\mathbf{R}^n) \quad \text{such that} \\ F(x,s) \le G(x)|s|^\delta + C|s|^{2^*} \qquad \forall s \in \mathbf{R} \text{ and for a.e. } x \in \mathbf{R}^n \ . \end{cases} \tag{12}$$

In Section 3 we will prove

Theorem 1 *Assume (6)-(12); then (2) admits a positive weak solution $\bar{u} \in E$.*

Let us turn to the problem (3): we assume that $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ is a (nontrivial) measurable function such that

$$|f(x,s)| \le C(|s| + |s|^p) \quad \text{for a.e. } (x,s) \in \mathbf{R}^n \times \mathbf{R} \ , \tag{13}$$

where C is a positive constant and $1 < p \le \frac{n+2}{n-2}$. Here we do not assume that $f(x,\cdot)$ is continuous: nevertheless, if we define $F(x,s) = \int_0^s f(x,t)dt$ we observe that F is a Carathéodory function which is locally Lipschitz with respect to the second variable. We also observe that the functional

$$\Psi(u) = \int_{\mathbf{R}^n} F(x,u)$$

[1] One could also consider the case $\delta = 2$: in such case one also needs $\|G\|_{n/2}$ to be sufficiently small.

is locally Lipschitz on E. Indeed, by (13), Hölder's inequality and the embedding $E \subset L^{p+1}$,

$$|\Psi(u) - \Psi(v)| \leq C(\|u\|_E, \|v\|_E)\|u - v\|_E \ ,$$

where $C(\|u\|_E, \|v\|_E) > 0$ depends only on $\max\{\|u\|_E, \|v\|_E\}$.
We impose to f the following additional assumptions

$$\lim_{\varepsilon \searrow 0} \ \text{esssup} \left\{ \left| \frac{f(x,s)}{s} \right| ; \ (x,s) \in \mathbf{R}^n \times (-\varepsilon, \varepsilon) \right\} = 0 \tag{14}$$

and there exists $\mu > 2$ such that

$$0 \leq \mu F(x,s) \leq s\underline{f}(x,s) \quad \text{for a.e. } (x,s) \in \mathbf{R}^n \times [0, +\infty) \ . \tag{15}$$

In Section 4 we will prove

Theorem 2 *Under hypotheses (9), (13)-(15), problem (3) has at least a positive solution in E.*

Remark. The couple of assumptions (11) (12) is equivalent to the couple (13) (14) in the sense that Theorems 1 and 2 hold under any one of these couples of assumptions. □

It seems not possible to use the above mentioned nonsmooth critical point theories to obtain an existence result for the quasilinear operator of (2) in the presence of a function f which is discontinuous with respect to the second variable; indeed, to prove that critical points of J (in the sense of [9, 10]) solve (2) in distributional sense, one needs, for all given $\varphi \in C_c^\infty$, the continuity of the map $u \mapsto J'(u)[\varphi]$, see [4]. Even if $J \notin C^1(E)$, we have at least $J \in C^1(W^{1,p} \cap E)$ for $p \geq \frac{3n}{n+1}$: this smoothness property in a finer topology is in fact the basic (hidden) tool used in Theorem 1.5 in [4]; however, one cannot prove the boundedness of the PS sequences in the $W^{1,p}$ norm. On the other hand, the theory developed in [6, 7] only applies to Lipschitz continuous functionals and therefore it does not allow to manage quasilinear operators as that in (2).

3 Proof of Theorem 1

Throughout this section we assume (6)-(12): by (6) and (8) we have

$$u \in E \Longrightarrow \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \in L^1(\mathbf{R}^n) \tag{16}$$

and therefore $J'(u)[u]$ can be written in integral form.
We first remark that positive solutions of (2) correspond to critical points of the functional J_+ defined by

$$J_+(u) := \frac{1}{2} \int_{\mathbf{R}^n} \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j u + \frac{1}{2} \int_{\mathbf{R}^n} b(x) u^2 - \int_{\mathbf{R}^n} F(x, u^+) \qquad \forall u \in E \ ,$$

where u^+ denotes the positive part of u, i.e. $u^+(x) = \max(u(x), 0)$.

Lemma 1 *Let $u \in E$ satisfy $J'_+(u)[\varphi] = 0$ for all $\varphi \in C_c^\infty$; then u is a weak positive solution of (2).*

For the proof of this result we refer to [8]; without loss of generality we can therefore suppose that

$$f(x,s) = 0 \qquad \forall s \le 0 \ , \quad \text{for a.e. } x \in \mathbf{R}^n$$

and, from now on, we make this assumption; for simplicity we denote J instead of J_+.

Let us establish the following boundedness criterion which applies, in particular, to PS sequences[2]:

Lemma 2 *Every sequence $\{u_m\} \subset E$ satisfying*

$$|J(u_m)| \le C_1 \qquad \text{and} \qquad |J'(u_m)[u_m]| \le C_2 \|u_m\|_E$$

is bounded in E.

Proof. Consider $\{u_m\} \subset E$ such that $|J(u_m)| \le C_1$, then by (10) we get

$$I_m := \frac{1}{2} \int_{\mathbf{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m - \frac{1}{\mu} \int_{\mathbf{R}^n} f(x,u_m) u_m + \frac{1}{2} \int_{\mathbf{R}^n} b(x) u_m^2 \le C_1 \ ;$$

by (16) we can evaluate $J'(u_m)[u_m]$ and by the assumptions we have

$$\Bigg| \int_{\mathbf{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m + \frac{1}{2} \int_{\mathbf{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m u_m + \\ - \int_{\mathbf{R}^n} f(x,u_m) u_m + \int_{\mathbf{R}^n} b(x) u_m^2 \Bigg| \le C_2 \|u_m\|_E \ .$$

Therefore, by (8) and computing $I_m - \frac{1}{\mu} J'(u_m)[u_m]$ we get

$$\frac{\mu - 2 - \gamma}{2\mu} \int_{\mathbf{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m + \frac{\mu - 2}{2\mu} \int_{\mathbf{R}^n} b(x) u_m^2 \le C_3 \|u_m\|_E + C_1 \ ;$$

by (7) this yields $C_4 > 0$ such that $C_4 \|u_m\|_E^2 \le C_3 \|u_m\|_E + C_1$ and the result follows. □

Let us denote by E_{loc} the space of functions u satisfying $\int_\omega (|Du|^2 + b(x)u^2) < \infty$ for all bounded open set $\omega \subset \mathbf{R}^n$ and by E^*_{loc} its dual space; we establish that the weak limit of a PS sequence solves (2):

Lemma 3 *Let $\{u_m\}$ be a bounded sequence in E satisfying*

$$\int_{\mathbf{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j \varphi + \frac{1}{2} \int_{\mathbf{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m \varphi = \langle \beta_m, \varphi \rangle \qquad \forall \varphi \in C_c^\infty$$

with $\{\beta_m\}$ converging in E^*_{loc} to some $\beta \in E^*_{\text{loc}}$. Then, up to a subsequence, $\{u_m\} \subset E$ converges in E_{loc} to some $u \in E$ satisfying

$$\int_{\mathbf{R}^n} \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j \varphi + \frac{1}{2} \int_{\mathbf{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u \varphi = \langle \beta, \varphi \rangle \qquad \forall \varphi \in C_c^\infty \ .$$

[2]We refer to [4, 9, 10] for the definition of PS sequences in our nonsmooth critical point framework.

Proof. As b is uniformly positive and locally bounded, for all bounded open set $\omega \subset \mathbf{R}^n$ we have

$$\int_\omega (|Du|^2 + b(x)u^2) < \infty \quad \Longleftrightarrow \quad \int_\omega (|Du|^2 + u^2) < \infty \ ;$$

therefore the proof is essentially the same as Lemma 3 in [8]: the basic tool is Theorem 2.1 in [3] which is used following the idea of [4]. □

The previous results allow to prove

Proposition 1 *Assume that $\{u_m\} \subset E$ is a PS sequence for J; then there exists $\bar{u} \in E$ such that (up to a subsequence)*
(i) $u_m \rightharpoonup \bar{u}$ in E
(ii) $u_m \to \bar{u}$ in $E_{\rm loc}$
(iii) $\bar{u} \geq 0$ and $\bar{u}$ solves (2) in weak sense.

Proof. By Lemma 2, the sequence $\{u_m\}$ is bounded and (i) follows. To obtain (ii) it suffices to apply Lemma 3 with $\beta_m = \alpha_m + f(x, u_m) - b(x)u_m \in E^*$ where $\alpha_m \to 0$ in E^*: indeed, if $u_m \rightharpoonup u$ in E, then $\beta_m \to \beta$ in $E^*_{\rm loc}$ with $\beta = f(x,u) - b(x)u$. Finally, (iii) follows from Lemmas 1 and 3. □

In order to build a PS sequence for the functional J we apply the mountain-pass Lemma [1] in the nonsmooth version [10], see also Theorem 2.1 in [2]: let us check that J has such a geometrical structure. First note that $J(0) = 0$; as the function F is superquadratic at $+\infty$, we may choose a nonnegative function e such that

$$e \in C^\infty_c \ , \quad e \geq 0 \quad \text{and} \quad J(te) < 0 \quad \forall t > 1 \ .$$

Moreover, it is easy to check that there exist $\rho, \beta > 0$ such that $\rho < \|e\|_E$ and $J(u) \geq \beta$ if $\|u\|_E = \rho$: indeed by (12) we infer

$$\int_{\mathbf{R}^n} F(x,u) \leq \|G\|_{q(\delta)} \|u\|^\delta_{2^*} + C\|u\|^{2^*}_{2^*} \ ;$$

hence, by (7) we have $J(u) \geq C_1\|u\|^2_E - C_2\|u\|^\delta_E - C_3\|u\|^{2^*}_E$ and the existence of ρ, β follows.
So, J has a mountain pass geometry; if we define the class

$$\Gamma := \{\gamma \in C([0,1];\ E);\ \gamma(0) = 0,\ \gamma(1) = e\} \tag{17}$$

and the minimax value

$$\alpha := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \ , \tag{18}$$

the existence of a PS sequence for J at level α follows by the results of [9, 10].
We have so proved

Proposition 2 *Let Γ and α be as in (17) (18); then J admits a PS sequence $\{u_m\}$ at level α.*

As we are on an unbounded domain, the problem lacks compactness and we cannot infer that the above PS sequence converges strongly; however, by using Proposition 1, the weak limit $\bar{u}$ of the PS sequence is a nonnegative solution of (2): the main problem is that it could be $\bar{u} \equiv 0$. To prove that this is not the case we make use of the following technical result:

Lemma 4 *There exist $p \in (2, 2^*)$ and $C > 0$ such that $\|u_m^+\|_p \geq C$.*

Proof. Using the relations $J'(u_m)[u_m] = o(1)$ and $J(u_m) = \alpha + o(1)$, by assumptions (8) and (10) we have

$$\begin{aligned} 2\alpha &= 2J(u_m) - J'(u_m)[u_m] + o(1) \\ &= \int_{\mathbf{R}^n} [f(x, u_m^+)u_m - 2F(x, u_m^+)] - \frac{1}{2}\int_{\mathbf{R}^n} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x, u_m) D_i u_m D_j u_m u_m + o(1) \\ &\leq \int_{\mathbf{R}^n} f(x, u_m^+)u_m + o(1) \ . \end{aligned}$$

Then, by (11), for all $\varepsilon > 0$ there exists $f_\varepsilon \in L^{\frac{2n}{n+2}}(\mathbf{R}^n)$ such that

$$2\alpha \leq \int_{\mathbf{R}^n} |f_\varepsilon(x) u_m^+(x)| + \varepsilon \|u_m^+\|_{2^*}^{2^*} \ :$$

as $\|u_m\|_{2^*}$ is bounded, one can choose $\varepsilon > 0$ so that

$$\alpha \leq \int_{\mathbf{R}^n} |f_\varepsilon(x) u_m^+(x)| \ . \tag{19}$$

Now take $r \in (\frac{2n}{n+2}, 2)$: then for all $\delta > 0$ there exist $f_\delta \in L^r$ and $f^\delta \in L^{\frac{2n}{n+2}}$ such that

$$f_\varepsilon = f_\delta + f^\delta \qquad \text{and} \qquad \|f^\delta\|_{\frac{2n}{n+2}} \leq \delta \ .$$

Then, by (19) and Hölder's inequality we infer

$$\alpha \leq \|f_\delta\|_r \|u_m^+\|_p + \delta \|u_m^+\|_{2^*}$$

where $p = \frac{r}{r-1}$; as $\|u_m\|_{2^*}$ is bounded, one can choose $\delta > 0$ so that

$$\frac{\alpha}{2} \leq \|f_\delta\|_r \|u_m^+\|_p$$

and the result follows. □

By the previous Lemma we deduce that $\{u_m^+\}$ does not converge strongly to 0 in L^p. Taking into account that $\|u_m^+\|_2$ and $\|\nabla u_m^+\|_2$ are bounded, by Lemma I.1 p. 231 in [12], we infer that the sequence $\{u_m^+\}$ "does not vanish" in L^2, i.e. there exists a sequence $\{y_m\} \subset \mathbf{R}^n$ and $C > 0$ such that

$$\int_{y_m + B_R} |u_m^+|^2 \geq C \tag{20}$$

for some R. We claim that the sequence $\{y_m\}$ is bounded: if not, up to a subsequence, it follows by (9) that

$$\int_{\mathbf{R}^n} b(x) u_m^2 \to +\infty$$

which contradicts $J(u_m) = \alpha + o(1)$. Therefore, by (20), there exists an open bounded set $\omega \subset \mathbf{R}^n$ such that

$$\int_\omega |u_m|^2 \geq C > 0 \ . \tag{21}$$

So, consider the PS sequence found in Proposition 2; by Proposition 1, it converges in the L^2_{loc} topology to some nonnegative function $\bar{u}$ which solves (2) in weak sense; finally, (21) entails $\bar{u} \not\equiv 0$.
The proof of Theorem 1 is complete.

4 Proof of Theorem 2

In this section we assume (9) and (13)-(15); moreover, we set $f(x,s) \equiv 0$ for $s \leq 0$.
To prove Theorem 2, it is sufficient to show that the functional I has a critical point $u_0 \in \mathcal{C}$, $\mathcal{C}$ being the cone of positive functions of E. Indeed,

$$\partial I(u) = -\Delta u + b(x)u - \partial\Psi(u) \quad \text{in } E^* ,$$

and, by Theorem 2.2 of [5] and Theorem 3 of [13], we have

$$\partial\Psi(u) \subset [\underline{f}(x,u(x)), \overline{f}(x,u(x))] \qquad \text{for a.e. } x \in \mathbf{R}^n ,$$

in the sense that if $w \in \partial\Psi(u)$ then

$$\underline{f}(x,u(x)) \leq w(x) \leq \overline{f}(x,u(x)) \qquad \text{for a.e. } x \in \mathbf{R}^n . \tag{22}$$

Thus, if u_0 is a critical point of I, then there exists $w \in \partial\Psi(u_0)$ such that

$$-\Delta u_0 + b(x)u_0 = w \qquad \text{in } E^* .$$

The existence of u_0 will be justified by a nonsmooth variant of the mountain-pass Lemma (see Theorem 1 of [15]), even if the PS condition is not fulfilled. More precisely, we verify the following geometric hypotheses:

$$I(0) = 0 \text{ and } \exists v \in E \text{ such that } I(v) \leq 0 \tag{23}$$

$$\exists \beta, \rho > 0 \quad \text{such that} \quad I \geq \beta \quad \text{on} \quad \{u \in E;\ \|u\|_E = \rho\} . \tag{24}$$

Verification of (23). It is obvious that $I(0) = 0$. For the second assertion we need

Lemma 5 *There exist two positive constants C_1 and C_2 such that*

$$f(x,s) \geq C_1 s^{\mu-1} - C_2 \qquad \textit{for a.e. } (x,s) \in \mathbf{R}^n \times [0,+\infty) . \tag{25}$$

Proof. From the definition we clearly have

$$\underline{f}(x,s) \leq f(x,s) \qquad \text{a.e. in } \mathbf{R}^n \times [0,+\infty) . \tag{26}$$

Then, by (15),

$$0 \leq \mu \underline{F}(x,s) \leq s\underline{f}(x,s) \qquad \text{for a.e. } (x,s) \in \mathbf{R}^n \times [0,+\infty) , \tag{27}$$

where

$$\underline{F}(x,s) = \int_0^s \underline{f}(x,t)dt .$$

By (27), there exist $R > 0$ and $K_1 > 0$ such that

$$\underline{F}(x,s) \geq K_1 s^{\mu} \qquad \text{for a.e. } (x,s) \in \mathbf{R}^n \times [R,+\infty) . \tag{28}$$

The inequality (25) follows now by (26), (27) and (28). $\square$

Verification of (23) continued. Choose $v \in C_c^\infty(\mathbf{R}^n) \setminus \{0\}$ so that $v \geq 0$ in $\mathbf{R}^n$; we obviously have

$$\int_{\mathbf{R}^n} (|Dv|^2 + b(x)v^2) < +\infty \ .$$

Then, by Lemma 5,

$$I(tv) = \frac{t^2}{2} \int_{\mathbf{R}^n} (|Dv|^2 + b(x)v^2) - \Psi(tv) \leq$$
$$\leq \frac{t^2}{2} \int_{\mathbf{R}^n} (|Dv|^2 + b(x)v^2) + C_2 t \int_{\mathbf{R}^n} v - C_1' t^\mu \int_{\mathbf{R}^n} v^\mu < 0 \ ,$$

for $t > 0$ large enough. $\square$

Verification of (24). First observe that (13) and (14) imply that, for any $\varepsilon > 0$, there exists a constant A_ε such that

$$|f(x,s)| \leq \varepsilon |s| + A_\varepsilon |s|^p \quad \text{for a.e. } (x,s) \in \mathbf{R}^n \times \mathbf{R} \ . \tag{29}$$

By (29) and Sobolev's embedding Theorem we have, for any $u \in E$

$$\Psi(u) \leq \frac{\varepsilon}{2} \int_{\mathbf{R}^n} u^2 + \frac{A_\varepsilon}{p+1} \int_{\mathbf{R}^n} |u|^{p+1} \leq \varepsilon C_3 \ \|u\|_E^2 + C_4 \ \|u\|_E^{p+1},$$

where ε is arbitrary and $C_4 = C_4(\varepsilon)$. Thus, by (9)

$$I(u) = \frac{1}{2} \int_{\mathbf{R}^n} (|Du|^2 + b(x)u^2) - \Psi(u) \geq C_5 \ \|u\|_E^2 - \varepsilon C_3 \ \|u\|_E^2 - C_4 \ \|u\|_E^{p+1} \geq \beta > 0 \ ,$$

for $\|u\|_E = \rho$, with ρ, ε and β sufficiently small positive constants. $\square$

Denote

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) \neq 0 \text{ and } I(\gamma(1)) \leq 0\}$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \, .$$

Set

$$\lambda_I(u) = \min_{\zeta \in \partial I(u)} \|\zeta\|_{E^*} \ .$$

Then, by Theorem 1 of [15], there exists a sequence $\{u_m\} \subset E$ such that

$$I(u_m) \to c \qquad \text{and} \qquad \lambda_I(u_m) \to 0 \ ; \tag{30}$$

since $I(|u|) \leq I(u)$ for all $u \in E$ we may assume that $\{u_m\} \subset \mathcal{C}$. So, there exists a sequence $\{w_m\} \subset \partial\Psi(u_m) \subset E^*$ such that

$$-\Delta u_m + b(x)u_m - w_m \to 0 \qquad \text{in } E^* \ . \tag{31}$$

Note that for all $u \in \mathcal{C}$, by (15) we have

$$\Psi(u) \leq \frac{1}{\mu} \int_{\mathbf{R}^n} u(x)\underline{f}(x, u(x)) \ .$$

Therefore, by (22), for every $u \in \mathcal{C}$ and any $w \in \partial\Psi(u)$,

$$\Psi(u) \leq \frac{1}{\mu} \int_{\mathbf{R}^n} u(x)w(x) \, .$$

Hence, if $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E^* and E, we have

$$\begin{aligned}
I(u_m) \quad &= \frac{\mu - 2}{2\mu} \int_{\mathbf{R}^n} (|Du_m|^2 + b(x)u_m^2) \\
&+ \frac{1}{\mu}\langle -\Delta u_m + bu_m - w_m, u_m\rangle + \frac{1}{\mu} \langle w_m, u_m\rangle - \Psi(u_m) \\
&\geq \frac{\mu - 2}{2\mu} \int_{\mathbf{R}^n} (|Du_m|^2 + b(x)u_m^2) + \frac{1}{\mu}\langle -\Delta u_m + bu_m - w_m, u_m\rangle \\
&\geq \frac{\mu - 2}{2\mu} \ \|u_m\|_E^2 - o(1) \ \|u_m\|_E \ .
\end{aligned}$$

This, together with (30), implies that the Palais-Smale sequence $\{u_m\}$ is bounded in E: thus, it converges weakly (up to a subsequence) in E and strongly in L^2_{loc} to some $u_0 \in \mathcal{C}$. Taking into account that $w_m \in \partial\Psi(u_m)$ for all m, that $u_m \rightharpoonup u_0$ in E and that there exists $w_0 \in E^*$ such that $w_m \rightharpoonup w_0$ in E^* (up to a subsequence), we infer that $w_0 \in \partial\Psi(u_0)$: this follows from the fact that the map $u \mapsto F(x, u)$ is compact from E into L^1. Moreover, if we take $\varphi \in C_c^\infty(\mathbf{R}^n)$ and let $\Omega := \mathrm{supp}\varphi$, then by (31) we get

$$\int_\Omega (Du_0 D\varphi + b(x)u_0\varphi - w_0\varphi) = 0 \ ;$$

as $w_0 \in \partial\Psi(u_0)$, by using (4) p.104 in [5] and by definition of $(-F)^0$, this implies

$$\int_\Omega (Du_0 D\varphi + b(x)u_0\varphi) + \int_\Omega (-F)^0(x, u_0; \varphi) \geq 0 \ .$$

By density, this hemivariational inequality holds for all $\varphi \in E$ and (5) follows; this means that u_0 solves problem (3).

It remains to prove that $u_0 \not\equiv 0$. If w_m is as in (31), then by (22) (recall that $u_m \in \mathcal{C}$) and (30) (for large m) we get

$$\frac{c}{2} \leq I(u_m) - \frac{1}{2} \ \langle -\Delta u_m + bu_m - w_m, u_m\rangle = \frac{1}{2} \ \langle w_m, u_m\rangle - \int_{\mathbf{R}^n} F(x, u_m) \leq \frac{1}{2} \int_{\mathbf{R}^n} u_m\overline{f}(x, u_m) \ . \quad (32)$$

Now, taking into account its definition, one deduces that $\overline{f}$ verifies (29), too. So, by (32), we obtain

$$\frac{c}{2} \leq \frac{1}{2} \int_{\mathbf{R}^n} (\varepsilon|u_m|^2 + A_\varepsilon|u_m|^{p+1}) = \frac{\varepsilon}{2} \ \|u_m\|_2^2 + \frac{A_\varepsilon}{2} \ \|u_m\|_{p+1}^{p+1} \ ;$$

hence, $\{u_m\}$ does not converge strongly to 0 in L^{p+1}. Frow now on, with the same arguments as in the proof of Theorem 1 (see after Lemma 4), we deduce that $u_0 \not\equiv 0$, which ends our proof.

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Perturbations of nonsmooth symmetric nonlinear eigenvalue problems

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Abstract. We consider a symmetric semilinear boundary value problem having infinitely many solutions. We prove that, if we perturb this problem in a non-symmetric way, then the number of solutions goes to infinity as the perturbation tends to zero. The growth conditions on the nonlinearities do not ensure the smoothness of the associated functional.

Perturbations des problèmes non-linéaires aux valeurs propres symétriques non réguliers

Résumé. *On considère un problème semi-linéaire symétrique avec une infinité de solutions. On montre que, si l'on perturbe ce problème d'une manière non-symétrique, alors le nombre de solutions devient de plus en plus grand lorsque la perturbation tend vers zéro. Les conditions de croissance sur les nonlinéarités ne garantissent pas la régularité de la fonctionnelle associée.*

Version française abrégée

Soit $\Omega \subset \mathbf{R}^N$ un ouvert borné. Pour $r > 0$ fixé arbitrairement on considère le problème suivant: trouver $(u, \lambda) \in H_0^1(\Omega) \times \mathbf{R}$ tel que

$$(1) \qquad \begin{cases} f(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda f(x,u) \quad \text{dans } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,. \end{cases}$$

On suppose que $f : \Omega \times \mathbf{R} \to \mathbf{R}$ est une fonction de Carathéodory avec les propriétés suivantes:

$(f1)$ $f(x,-s) = -f(x,s)$, p.p. sur Ω et pour chaque $s \in \mathbf{R}$;

$(f2)$ ils existent $a \in L^1(\Omega)$, $b \in \mathbf{R}$ et $0 \leq p < \frac{2N}{N-2}$ (si $N > 2$) tels que

$$0 < sf(x,s) \leq a(x) + b|s|^p\,, \qquad F(x,s) \leq a(x) + b|s|^p\,,$$

p.p. sur Ω et pour chaque $s \in \mathbf{R} \setminus \{0\}$, où $F(x,s) = \int_0^s f(x,t)dt$;

$(f3)$ $\sup\limits_{|s|\leq t} |f(x,s)| \in L^1_{loc}(\Omega)$, pour chaque $t > 0$.

THÉORÈME 1. - *Supposons que les conditions $(f1)-(f3)$ soient satisfaites. Alors le problème (1) admet une suite $(\pm u_n, \lambda_n)$ de solutions distinctes.*

Ensuite notre objectif est d'analyser le problème perturbé

$$\begin{cases} f(x,u),\ g(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda\,(f(x,u) + g(x,u)) \quad \text{dans } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,, \end{cases} \tag{2}$$

où $g : \Omega \times \mathbf{R} \to \mathbf{R}$ est une fonction de Carathéodory qui n'est pas nécessairement impaire par rapport à la seconde variable. On suppose quand même que g satisfait

$(g1)$ $0 < sg(x,s) \leq a(x) + b|s|^p$ p.p. sur Ω et pour chaque $s \in \mathbf{R} \setminus \{0\}$;

$(g2)$ $\sup\limits_{|s|\leq t} |g(x,s)| \in L^1_{loc}(\Omega)$, pour chaque $t > 0$;

$(g3)$ $G(x,s) \leq C_g\,(1+|s|^p)$, p.p. sur Ω et pour chaque $s \in \mathbf{R}$, avec $C_g > 0$, où $G(x,s) = \int_0^s g(x,t)dt$.

On démontre que le nombre de solutions du problème perturbé (2) devient de plus en plus grand si la perturbation est assez petite, dans un sens précisé ultérieurement. Plus précisement, on a

THÉORÈME 2. - *Supposons que les conditions $(f1)-(f3)$ et $(g1)-(g3)$ soient satisfaites. Alors, pour chaque entier $n \geq 1$, il existe $\varepsilon_n > 0$ tel que le problème (2) admet au moins n solutions distinctes si g est une fonction telle que la condition $(g3)$ soit satisfaite pour $C_g = \varepsilon_n$.*

La preuve des Théorèmes 1 et 2 repose sur un argument variationnel. D'abord on pose

$$S_r = \left\{ u \in H^1_0(\Omega) : \int_\Omega |Du|^2 dx = r^2 \right\}$$

et on étudie les points critiques sur S_r de la fonctionnelle continue et paire $I : H^1_0(\Omega) \to \mathbf{R}$ définie par

$$I(u) = -\int_\Omega F(x,u)dx\,.$$

REMARQUE 1. - *Si* $(f2)$, $(f3)$ *sont remplacées par la condition standarde* $0 < sf(x,s) \leq a_1(x)|s| + b|s|^p$ *avec* $a_1 \in L^{\frac{2N}{N+2}}(\Omega)$, *alors* I *est de classe* C^1 *et le Théorème 1 se trouve dans [8, Theorem 8.17]. Avec notres hypothèses,* f *peut avoir la forme* $f(x,s) = \alpha(x)\gamma(s)$ *avec* $\alpha \in L^1(\Omega)$, $\alpha \geq 0$, $\gamma \in C_c(\mathbf{R})$, γ *impaire et* $s\gamma(s) \geq 0$ *pour chaque* $s \in \mathbf{R}$. *Dans ce cas là,* I *est bien sûr continue, mais pas localement Lipschitz.*

REMARQUE 2. - *Lorsque* f *et* g *satisfont la condition standarde qu'on vient de mentionner, résultats du type du Théorème 2 sont bien classiques (voir par example Krasnoselskii [7]). Des résultats de perturbation, plutôt différents des nôtres, où le problème perturbé admet encore une infinité de solutions, peuvent être trovés dans [8, 9]. Dans un cadre non régulier, un résultat dans la ligne du Théorème 2 a été démontré dans [4] lorsque* f *et* g *satisfont la condition standarde, mais la fonction* u *est contrainte par un obstacle, de sorte que l'équation se transforme dans une inéquation variationnelle.*

Let $\Omega \subset \mathbf{R}^N$ be a bounded open set. For some fixed $r > 0$, consider the problem: find $(u, \lambda) \in H_0^1(\Omega) \times \mathbf{R}$ such that

$$
(1) \qquad \begin{cases} f(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda f(x,u) \quad \text{in } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,, \end{cases}
$$

where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that the following conditions hold:

$(f1)$ $f(x,-s) = -f(x,s)$, for a.e. $x \in \Omega$ and every $s \in \mathbf{R}$;

$(f2)$ there exist $a \in L^1(\Omega)$, $b \in \mathbf{R}$ and $0 \leq p < \frac{2N}{N-2}$ (if $N > 2$) such that

$$
0 < sf(x,s) \leq a(x) + b|s|^p\,, \qquad F(x,s) \leq a(x) + b|s|^p\,,
$$

for a.e. $x \in \Omega$ and every $s \in \mathbf{R} \setminus \{0\}$, where $F(x,s) = \int_0^s f(x,t)dt$;

$(f3)$ $\sup\limits_{|s| \leq t} |f(x,s)| \in L^1_{loc}(\Omega)$, for every $t > 0$.

We notice that, if $N = 1$, then in condition $(f2)$ the term $b|s|^p$ can be substituted by any continuous function $\varphi(s)$ of s, while, if $N = 2$, the same term can be substituted by $\exp(\varphi(s))$, with $\varphi(s)s^{-2} \to 0$ as $|s| \to \infty$.

THEOREM 1. - *Assume that hypotheses* $(f1) - (f3)$ *hold. Then Problem (1) admits a sequence* $(\pm u_n, \lambda_n)$ *of distinct solutions.*

Then we want to study what happens when the energy functional is subjected to a perturbation which destroys the symmetry.

Consider the problem: find $(u, \lambda) \in H_0^1(\Omega) \times \mathbf{R}$ such that

$$
(2) \qquad \begin{cases} f(x,u),\, g(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda\,(f(x,u) + g(x,u)) \quad \text{in } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,, \end{cases}
$$

where $g : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function. We make no symmetry assumption on g, but we impose only

$(g1)$ $0 < sg(x,s) \le a(x) + b|s|^p$ for a.e. $x \in \Omega$ and every $s \in \mathbf{R} \setminus \{0\}$;

$(g2)$ $\sup\limits_{|s| \le t} |g(x,s)| \in L^1_{loc}(\Omega)$, for every $t > 0$;

$(g3)$ $G(x,s) \le C_g\,(1 + |s|^p)$, for a.e. $x \in \Omega$ and every $s \in \mathbf{R}$, for some $C_g > 0$, where $G(x,s) = \int_0^s g(x,t)dt$.

Our second result shows that the number of solutions of Problem (2) becomes greater and greater, as the perturbation tends to zero. More precisely we have

THEOREM 2. - *Assume that hypotheses $(f1) - (f3)$ and $(g1) - (g3)$ hold. Then, for every positive integer n, there exists $\varepsilon_n > 0$ such that Problem (2) admits at least n distinct solutions, provided that $(g3)$ holds for $C_g = \varepsilon_n$.*

We will prove Theorems 1 and 2 by a variational argument. First we set

$$
S_r = \left\{ u \in H_0^1(\Omega) : \int_\Omega |Du|^2 dx = r^2 \right\}
$$

and we study the critical points on S_r of the even continuous functional $I : H_0^1(\Omega) \to \mathbf{R}$ defined by

$$
I(u) = -\int_\Omega F(x,u)dx\,.
$$

REMARK 1. - *If $(f2)$, $(f3)$ are substituted by the more standard condition $0 < sf(x,s) \le a_1(x)|s| + b|s|^p$ with $a_1 \in L^{\frac{2N}{N+2}}(\Omega)$, then I is of class C^1 and Theorem 1 can be found in [8, Theorem 8.17]. Under our assumptions, f could have the form $f(x,s) = \alpha(x)\gamma(s)$ with $\alpha \in L^1(\Omega)$, $\alpha \ge 0$, $\gamma \in C_c(\mathbf{R})$, γ odd and $s\gamma(s) \ge 0$ for any $s \in \mathbf{R}$. In such a case, I is clearly continuous, but not locally Lipschitz.*

REMARK 2. - *When f and g are subjected to the standard condition we have mentioned, results like Theorem 2 go back to Krasnoselskii [7]. For perturbation results, quite different from ours, where the perturbed problem still has infinitely many solutions, we refer the reader to [8, 9]. In a nonsmooth setting, a result in the line of Theorem 2 has been proved in [4] when f and g satisfy the standard condition, but the function u is subjected to an obstacle, so that the equation becomes a variational inequality.*

From $(f2)$ it easily follows that $I(u) < 0$ and that $\sup I_r(u) = 0$, where $I_r = I_{|S_r}$.
Since I is only continuous, we will apply the nonsmooth techniques developed in [1, 3, 4, 5]. In the following, we will adopt the notations of such papers.

LEMMA 1. - *The following facts hold:*

(*a*) *if* $u \in S_r$ *satisfies* $|dI_r|(u) < +\infty$, *then* $f(x,u) \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$ *and there exists* $\mu \in \mathbf{R}$ *such that*

$$\|\mu\Delta u + f(x,u)\|_{H^{-1}} \leq |dI_r|(u);$$

(*b*) *the functional* I_r *satisfies* $(PS)_c$ *for any* $c < 0$;

(*c*) *if* $u \in S_r$ *is a critical point of* I_r, *then there exists* $\lambda > 0$ *such that* (u, λ) *is a solution of Problem (1).*

Proof. -
(*a*) Set also

$$I_{r,est}(w) = \begin{cases} I(w) & \text{if } w \in S_r\,, \\ \\ +\infty & \text{if } w \in H^1_0(\Omega) \setminus S_r\,. \end{cases}$$

Then it is immediately seen that $|dI_{r,est}|(u) = |dI_r|(u)$, where we are using the weak slope introduced in [5] (see also [1, Definition 2.1]). By [1, Theorem 4.13] there exists $\alpha \in \partial I_{r,est}(u)$ with $\|\alpha\|_{H^{-1}} \leq |dI_{r,est}|(u)$, where ∂ stands for the subdifferential introduced in [1, Definition 4.1]. Taking into account $(f2)$, we deduce from [6, Theorem 3.3] that

$$I^0(u;0) \leq 0\,, \qquad I^0(u;2u) \leq -2\int_\Omega f(x,u)u\,dx < +\infty\,.$$

Actually, the same proof shows a stronger fact, namely that

$$\overline{I}^0(u;0) \leq 0\,, \qquad \overline{I}^0(u;2u) \leq -2\int_\Omega f(x,u)u\,dx < +\infty\,.$$

Therefore we can apply [1, Corollary 5.10], obtaining $\beta \in \partial I(u)$ and $\mu \in \mathbf{R}$ with $\alpha = \beta - \mu\Delta u$. From [6, Theorems 3.3 and 2.25] we conclude that $f(x,u) \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$ and $\beta = -f(x,u)$. Then (*a*) easily follows.
(*b*) Let $c < 0$ and let (u_n) be a $(PS)_c$-sequence for I_r. By the previous point, we have $f(x,u_n) \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$ and there exists a sequence (μ_n) in $\mathbf{R}$ with

$$\|\mu_n\Delta u_n + f(x,u_n)\|_{H^{-1}} \to 0\,.$$

Up to a subsequence, (u_n) is convergent to some u weakly in $H^1_0(\Omega)$ and a.e. From $(f2)$ it follows $I(u) = c < 0$, hence $u \neq 0$. Again by $(f2)$ and Lebesgue's Theorem, we deduce that

$$0 < \int_\Omega f(x,u)u\,dx = \lim_n \int_\Omega f(x,u_n)u_n\,dx = \lim_n \mu_n \int_\Omega |Du_n|^2 dx\,.$$

Therefore, up to a further subsequence, (μ_n) is convergent to some $\mu > 0$ and

$$\left\|\Delta u_n + \frac{1}{\mu} f(x,u_n)\right\|_{H^{-1}} \to 0\,.$$

From [6, Lemma 4.8] we deduce that (u_n) is precompact in $H^1_0(\Omega)$ and (*b*) follows.
(*c*) Arguing as in (*b*), we find that $f(x,u) \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$ and that there exists $\mu > 0$ with $\mu\Delta u + f(x,u) = 0$. Then the assertion easily follows. ■

LEMMA 2. - *There exists a sequence (b_n) of essential values of I_r strictly increasing to* 0.

Proof. - We will adapt some arguments from [4] to our concrete situation. Let $\psi :]-\infty, 0[\to \mathbf{R}$ be an increasing diffeomorphism. From Lemma 1 it follows that $\psi \circ I_r$ satisfies $(PS)_c$ for every $c \in \mathbf{R}$. Then by [2, Theorem 1.4.13] we have that $\{u \in S_r : \psi \circ I_r(u) \leq b\}$ has finite genus for every $b \in \mathbf{R}$. If (c_n) is the sequence defined as in [4, Theorem 2.12] with respect to $\psi \circ I_r$, it follows that $c_n \to +\infty$ as $n \to \infty$. Therefore there exists a sequence (b'_n) of essential values of $\psi \circ I_r$ strictly increasing to $+\infty$. Then $b_n = \psi^{-1}(b'_n)$ has the required properties. ■

Proof of Theorem 1. - Combining Lemma 1 with [4, Theorem 2.10], we deduce that each b_n is a critical value of I_r. Again from Lemma 1 we conclude that there exists a sequence $(\pm u_n, \lambda_n)$ of solutions of Problem 1 with $I(u_n) = b_n$ strictly increasing to 0. ■

Now we introduce the continuous functional $J_r : S_r \to \mathbf{R}$ defined by

$$J(u) = I(u) - \int_\Omega G(x,u)dx\,.$$

LEMMA 3. - *For every $\eta > 0$, there exists $\varepsilon > 0$ such that $\sup\limits_{u \in S_r} |I_r(u) - J_r(u)| < \eta$, provided that $(g3)$ holds for $C_g = \varepsilon$.*

Proof. - By Sobolev inclusions, we have

$$0 \leq I_r(u) - J_r(u) = \int_\Omega G(x,u)dx \leq C_g \int_\Omega (1 + |u|^p)dx < \eta, \qquad \text{for any } u \in S_r,$$

if g is chosen as in the hypothesis. ■

Proof of Theorem 2. - As in the proof of Theorem 1, let us consider a strictly increasing sequence (b_n) of essential values of I_r such that $b_n \to 0$ as $n \to \infty$. Given $n \geq 1$, take some $\delta > 0$ with $b_n + \delta < 0$ and $2(b_j - b_{j-1}) < \delta$ for $j = 2, \ldots, n$. We apply [4, Theorem 2.6] to I_r and J_r. So, for any $j = 1, \ldots, n$, there exists $\eta_j > 0$ such that $\sup\limits_{u \in S_r} |I_r(u) - J_r(u)| < \eta_j$ implies the existence of an essential value $c_j \in]b_j - \delta, b_j + \delta[$ of J_r. We now apply Lemma 3 for $\eta = \min\{\eta_1, \ldots, \eta_n\}$. Thus we obtain $\varepsilon_n > 0$ such that $\sup\limits_{u \in S_r} |I_r(u) - J_r(u)| < \eta$, if $(g3)$ holds with $C_g = \varepsilon_n$. It follows that J_r has at least n distinct essential values $c_1, \ldots, c_n$ in the interval $]-\infty, 0[$.

Now Lemma 1 can be clearly adapted to the functional J_r. Then we find $u_1, \ldots, u_n \in S_r$ and $\lambda_1, \ldots, \lambda_n > 0$ such that each (u_j, λ_j) is a solution of Problem 2 with $J_r(u_j) = c_j$. ■

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Multiple solutions of hemivariational inequalities with area-type term

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Abstract

Hemivariational inequalities containing both an area-type and a non-locally Lipschitz term are considered. Multiplicity results are obtained by means of techniques of nonsmooth critical point theory.

1 Introduction

The theory of variational inequalities appeared in the middle 60's in connection with the notion of subdifferential in the sense of Convex Analysis (see e.g. [6, 22, 33] for the main aspects of this theory). All the inequality problems treated to the beginning 80's were related to convex energy functionals and therefore strictly connected to monotonicity: for instance, only monotone (possibly multivalued) boundary conditions and stress-strain laws could be studied.

Nonconvex inequality problems first appeared in [35] in the setting of Global analysis and were related to the subdifferential introduced in [17] (see A. Marino [34] for a survey of the developments in this direction).

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In the setting of Continuum mechanics, P. D. PANAGIOTOPOULOS started the study of nonconvex and nonsmooth potentials by using Clarke's subdifferential for locally Lipschitz functionals. Due to the lack of convexity, new types of inequality problems, called hemivariational inequalities, have been generated. Roughly speaking, mechanical problems involving nonmonotone stress-strain laws or boundary conditions derived by nonconvex superpotentials lead to hemivariational inequalities. We refer the reader to [41, 42] for the main aspects of this theory.

A typical feature of nonconvex problems is that, while in the convex case the stationary variational inequalities give rise to minimisation problems for the potential or for the energy, in the nonconvex case the problem of the stationarity of the potential emerges and therefore it becomes reasonable to expect results also in the line of critical point theory.

For hemivariational inequalities, several contributions have been recently obtained by techniques of nonsmooth critical point theory (see [5, 23, 25, 26, 27, 28, 38, 39, 40, 43] and references therein). The associated functional f is typically of the form $f = f_0 + f_1$, where f_0 is the principal part satisfying some standard coerciveness condition and f_1 is locally Lipschitz. In such a setting, the main abstract tool is constituted by the nonsmooth critical point theory developed in [12] for locally Lipschitz functionals.

The aim of our paper is to obtain existence and multiplicity results for hemivariational inequalities associated with functionals which come from the relaxation of, say,

$$f(u) = \int_\Omega \sqrt{1+|Du|^2}\,dx + \int_\Omega G(x,u)\,dx\,, \qquad u \in W_0^{1,1}(\Omega;\mathbf{R}^N),\ \Omega \text{ open in } \mathbf{R}^n,\ n \geq 2\,.$$

The first feature is that the functional f does not satisfy the Palais-Smale condition in $BV(\Omega;\mathbf{R}^N)$, the natural domain of f, as it is already known in the case of equations (see e.g. [36]). Therefore we extend f to $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ with value $+\infty$ outside $BV(\Omega;\mathbf{R}^N)$. This larger space is better behaved for the compactness properties, but the nonsmoothness of the functional increases. The second feature is that the assumptions we impose on G imply the second term of f to be continuous on $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$, but not locally Lipschitz. More precisely, the function $\{s \mapsto G(x,s)\}$ is supposed to be locally Lipschitz for a.e. $x \in \Omega$, but the growth conditions we impose do not ensure the corresponding property for the integral on $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$. Because of these facts, we will take advantage of the nonsmooth techniques developed in [7, 16, 19], which have been already applied in the setting of equations (see [8, 9, 10, 15, 18, 20, 21, 23, 36, 37] and references therein) and turn out to be suitable also for our setting.

In section 2 we recall the main tools we will need, while in section 3 we prove some general results for a class of lower semicontinuous functionals $f : L^p(\Omega;\mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$. In section 4 we show that the area-type integrals fall into the class considered in section 3. By the way, we also prove a relation between the convergence in the so-called intermediate topologies of $BV(\Omega;\mathbf{R}^N)$ and the convergence in $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ (see Theorem 4.10), which seems to be new. Finally, in sections 5 and 6 we apply the general setting of section 3 to obtain multiplicity results of Clark and Ambrosetti-Rabinowitz type. Of course, we believe that our approach could be equally applied to other situations with different geometries.

2 Recalls of nonsmooth analysis

Let X be a metric space endowed with the metric d and let $f : X \to \overline{\mathbf{R}}$ be a function. We denote by $\mathrm{B}_r(u)$ the open ball of centre u and radius r and we set

$$\mathrm{epi}\,(f) = \{(u,\lambda) \in X \times \mathbf{R} :\ f(u) \le \lambda\}\ .$$

In the following, $X \times \mathbf{R}$ will be endowed with the metric

$$d\left((u,\lambda),(v,\mu)\right) = \left(d(u,v)^2 + (\lambda-\mu)^2\right)^{\frac{1}{2}}$$

and $\mathrm{epi}\,(f)$ with the induced metric.

Definition 2.1 *For every $u \in X$ with $f(u) \in \mathbf{R}$, we denote by $|df|\,(u)$ the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : (\mathrm{B}_\delta\,(u,f(u)) \cap \mathrm{epi}\,(f)) \times [0,\delta] \to X$$

satisfying

$$d(\mathcal{H}((w,\mu),t),w) \le t\,, \qquad f(\mathcal{H}((w,\mu),t)) \le \mu - \sigma t\,,$$

whenever $(w,\mu) \in \mathrm{B}_\delta\,(u,f(u)) \cap \mathrm{epi}\,(f)$ and $t \in [0,\delta]$.

The extended real number $|df|\,(u)$ is called the weak slope *of f at u.*

The above notion has been introduced in [19], following an equivalent approach. When f is continuous, it has been independently introduced also in [32], while a variant has been considered in [30, 31]. The version we have recalled here is taken from [7].

Now, according to [17], we define a function $\mathcal{G}_f : \mathrm{epi}\,(f) \to \mathbf{R}$ by $\mathcal{G}_f(u,\lambda) = \lambda$. Of course, $\mathcal{G}_f$ is Lipschitz continuous of constant 1.

Proposition 2.2 *For every $u \in X$ with $f(u) \in \mathbf{R}$, we have $f(u) = \mathcal{G}_f(u,f(u))$ and*

$$|df|\,(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|\,(u,f(u))}{\sqrt{1 - |d\mathcal{G}_f|\,(u,f(u))^2}} & \text{if } |d\mathcal{G}_f|\,(u,f(u)) < 1\,, \\ +\infty & \text{if } |d\mathcal{G}_f|\,(u,f(u)) = 1\,. \end{cases}$$

Proof. See [7, Proposition 2.3]. ■

The previous proposition allows us to reduce, at some extent, the study of the general function f to that of the continuous function $\mathcal{G}_f$.

Definition 2.1 can be simplified, when f is continuous.

Proposition 2.3 *Let $f : X \to \mathbf{R}$ be continuous. Then $|df|\,(u)$ is the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : \mathrm{B}_\delta\,(u) \times [0,\delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t),w) \le t\,, \qquad f(\mathcal{H}(w,t)) \le f(w) - \sigma t\,, \tag{2.4}$$

whenever $w \in \mathrm{B}_\delta\,(u)$ and $t \in [0,\delta]$.

Proof. See [7, Proposition 2.2]. ■

We need also, in a particular case, the notion of equivariant weak slope (see e.g. [10] for the general definition).

Definition 2.5 *Let X be a normed space and $f : X \to \overline{\mathbf{R}}$ an even function with $f(0) < +\infty$. For every $(0, \lambda) \in \mathrm{epi}\,(f)$ we denote by $|d_{\mathbf{Z}_2}\mathcal{G}_f|\,(0,\lambda)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : (\mathrm{B}_\delta\,(0,\lambda) \cap \mathrm{epi}\,(f)) \times [0,\delta] \to \mathrm{epi}\,(f)$$

satisfying

$$d\,(\mathcal{H}((w,\mu),t),(w,\mu)) \le t\,, \qquad \mathcal{H}_2((w,\mu),t) \le \mu - \sigma t\,,$$
$$\mathcal{H}_1((-w,\mu),t) = -\mathcal{H}_1((w,\mu),t)\,,$$

whenever $(w,\mu) \in \mathrm{B}_\delta\,(0,\lambda) \cap \mathrm{epi}\,(f)$ and $t \in [0,\delta]$.

Remark 2.6 *In Proposition 2.3, if there exist $\varrho > 0$ and a continuous map $\mathcal{H}$ satisfying*

$$d(\mathcal{H}(w,t),w) \le \varrho t\,, \qquad f(\mathcal{H}(w,t)) \le f(w) - \sigma t\,,$$

instead of (2.4), we can deduce that $|df|\,(u) \ge \sigma/\varrho$.

A similar remark applies to Definition 2.5.

By means of the weak slope, we can now introduce the two main notions of critical point theory.

Definition 2.7 *We say that $u \in X$ is* a (lower) critical point *of f, if $f(u) \in \mathbf{R}$ and $|df|\,(u) = 0$. We say that $c \in \mathbf{R}$ is* a (lower) critical value *of f, if there exists a (lower) critical point $u \in X$ of f with $f(u) = c$.*

Definition 2.8 *Let $c \in \mathbf{R}$. A sequence (u_h) in X is said to be* a Palais-Smale sequence at level c *($(PS)_c$*–sequence*, for short) for f, if $f(u_h) \to c$ and $|df|\,(u_h) \to 0$.*

We say that f satisfies the Palais-Smale condition at level c *($(PS)_c$, for short), if every $(PS)_c$–sequence (u_h) for f admits a convergent subsequence (u_{h_k}) in X.*

The main feature of the weak slope is that it allows to prove natural extensions of the classical critical point theory for general continuous functions defined on complete metric spaces. Moreover, one can try to reduce the study of a lower semicontinuous function f to that of the continuous function $\mathcal{G}_f$. Actually, Proposition 2.2 suggests to exploit the bijective correspondence between the set where f is finite and the graph of f. This approach can be successful, if we can ensure that the remaining part of epi (f) does not carry much information. The next notion turns out to be useful for this purpose.

Definition 2.9 *Let $c \in \mathbf{R}$. We say that f satisfies condition $(epi)_c$, if there exists $\varepsilon > 0$ such that*

$$\inf\{|d\mathcal{G}_f|\,(u,\lambda) :\ f(u) < \lambda,\ |\lambda - c| < \varepsilon\} > 0\,.$$

The next two results may help in dealing with condition $(epi)_c$.

Proposition 2.10 *Let* $(u, \lambda) \in \mathrm{epi}\,(f)$. *Assume that there exist* $\varrho, \sigma, \delta, \varepsilon > 0$ *and a continuous map*

$$\mathcal{H} : \{w \in \mathrm{B}_\delta\,(u) : f(w) < \lambda + \delta\} \times [0, \delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t), w) \leq \varrho t\,, \qquad f(\mathcal{H}(w,t)) \leq \max\{f(w) - \sigma t, \lambda - \varepsilon\}$$

whenever $w \in \mathrm{B}_\delta\,(u)$, $f(w) < \lambda + \delta$ *and* $t \in [0, \delta]$.

Then we have

$$|d\mathcal{G}_f|\,(u, \lambda) \geq \frac{\sigma}{\sqrt{\varrho^2 + \sigma^2}}\,.$$

If moreover X *is a normed space,* f *is even,* $u = 0$ *and* $\mathcal{H}(-w, t) = -\mathcal{H}(w, t)$, *then we have*

$$|d_{\mathbf{Z}_2}\mathcal{G}_f|\,(0, \lambda) \geq \frac{\sigma}{\sqrt{\varrho^2 + \sigma^2}}\,.$$

Proof. Let $\delta' \in]0, \delta]$ be such that $\delta' + \sigma\delta' \leq \varepsilon$ and let

$$\mathcal{K} : (\mathrm{B}_{\delta'}\,(u, \lambda) \cap \mathrm{epi}\,(f)) \times [0, \delta'] \to \mathrm{epi}\,(f)$$

be defined by $\mathcal{K}((w, \mu), t) = (\mathcal{H}(w, t), \mu - \sigma t)$. If $(w, \mu) \in \mathrm{B}_{\delta'}\,(u, \lambda) \cap \mathrm{epi}\,(f)$ and $t \in [0, \delta']$, we have

$$\lambda - \varepsilon \leq \lambda - \delta' - \sigma\delta' < \mu - \sigma t\,, \qquad f(w) - \sigma t \leq \mu - \sigma t\,,$$

hence

$$f(\mathcal{H}(w,t)) \leq \max\{f(w) - \sigma t, \lambda - \varepsilon\} \leq \mu - \sigma t\,.$$

Therefore $\mathcal{K}$ actually takes its values in $\mathrm{epi}\,(f)$. Furthermore, it is

$$d(\mathcal{K}((w, \mu), t), (w, \mu)) \leq \sqrt{\varrho^2 + \sigma^2}\, t\,,$$

$$\mathcal{G}_f(\mathcal{K}((w, \mu), t)) = \mu - \sigma t = \mathcal{G}_f(w, \mu) - \sigma t\,.$$

Taking into account Proposition 2.3 and Remark 2.6, the first assertion follows.

In the symmetric case, $\mathcal{K}$ automatically satisfies the further condition required in Definition 2.5. ∎

Corollary 2.11 *Let* $(u, \lambda) \in \mathrm{epi}\,(f)$ *with* $f(u) < \lambda$. *Assume that for every* $\varrho > 0$ *there exist* $\delta > 0$ *and a continuous map*

$$\mathcal{H} : \{w \in \mathrm{B}_\delta\,(u) : f(w) < \lambda + \delta\} \times [0, \delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t), w) \leq \varrho t\,, \qquad f(\mathcal{H}(w,t)) \leq f(w) + t(f(u) - f(w) + \varrho)$$

whenever $w \in \mathrm{B}_\delta\,(u)$, $f(w) < \lambda + \delta$ *and* $t \in [0, \delta]$.

Then we have $|d\mathcal{G}_f|\,(u, \lambda) = 1$. *If moreover* X *is a normed space,* f *is even,* $u = 0$ *and* $\mathcal{H}(-w, t) = -\mathcal{H}(w, t)$, *then we have* $|d_{\mathbf{Z}_2}\mathcal{G}_f|\,(0, \lambda) = 1$.

Proof. Let $\varepsilon > 0$ with $\lambda - 2\varepsilon > f(u)$, let $0 < \varrho < \lambda - f(u) - 2\varepsilon$ and let δ and $\mathcal{H}$ be as in the hypothesis. By reducing δ, we may also assume that

$$\delta \leq 1\,, \qquad \delta(|\lambda - 2\varepsilon| + |f(u) + \varrho|) \leq \varepsilon\,.$$

Now consider $w \in \mathrm{B}_\delta(u)$ with $f(w) < \lambda + \delta$ and $t \in [0, \delta]$. If $f(w) \leq \lambda - 2\varepsilon$, we have

$$\begin{aligned} f(w) + t(f(u) - f(w) + \varrho) &= (1-t)f(w) + t(f(u) + \varrho) \leq \\ &\leq (1-t)(\lambda - 2\varepsilon) + t(f(u) + \varrho) \leq \\ &\leq \lambda - 2\varepsilon + t|\lambda - 2\varepsilon| + t|f(u) + \varrho| \leq \lambda - \varepsilon\,, \end{aligned}$$

while, if $f(w) > \lambda - 2\varepsilon$, we have

$$f(w) + t(f(u) - f(w) + \varrho) \leq f(w) - (\lambda - f(u) - 2\varepsilon - \varrho)t\,.$$

In any case it follows

$$f(\mathcal{H}(w,t)) \leq \max\left\{f(w) - (\lambda - f(u) - 2\varepsilon - \varrho)t, \lambda - \varepsilon\right\}\,.$$

From Proposition 2.10 we get

$$|d\mathcal{G}_f|\,(u, \lambda) \geq \frac{\lambda - f(u) - 2\varepsilon - \varrho}{\sqrt{\varrho^2 + (\lambda - f(u) - 2\varepsilon - \varrho)^2}}$$

and the first assertion follows by the arbitrariness of ϱ.

The same proof works also in the symmetric case. ∎

Now we recall two critical point theorems we will apply later. The first one is an adaptation of a result of D. C. Clark (see [13] and [44, Theorem 9.1]) to our setting.

Theorem 2.12 *Let X be a Banach space and $f : X \to \mathbf{R} \cup \{+\infty\}$ an even lower semicontinuous function. Assume that*

(a) *f is bounded from below;*

(b) *for every $c < f(0)$, the function f satisfies $(PS)_c$ and $(epi)_c$;*

(c) *there exist $k \geq 1$ and an odd continuous map $\psi : S^{k-1} \to X$ such that*

$$\sup\left\{f(\psi(x)) : x \in S^{k-1}\right\} < f(0)\,,$$

where S^{k-1} denotes the unit sphere in $\mathbf{R}^k$.

Then f admits at least k pairs $(u_1, -u_1), \ldots, (u_k, -u_k)$ of critical points with $f(u_j) < f(0)$.

Proof. See [20, Theorem 2.5]. ∎

The next result is an adaptation of the classical Theorem of Ambrosetti-Rabinowitz [1, 44, 48].

Theorem 2.13 *Let X be a Banach space and $f : X \to \mathbf{R} \cup \{+\infty\}$ an even lower semicontinuous function. Assume that there exists a strictly increasing sequence (V_h) of finite-dimensional subspaces of X with the following properties:*

(*a*) *there exist a closed subspace Z of X, $\varrho > 0$ and $\alpha > f(0)$ such that $X = V_0 \oplus Z$ and*

$$\forall u \in Z : \ \|u\| = \varrho \implies f(u) \geq \alpha ;$$

(*b*) *there exists a sequence (R_h) in $]\varrho, +\infty[$ such that*

$$\forall u \in V_h : \ \|u\| \geq R_h \implies f(u) \leq f(0) ;$$

(*c*) *for every $c \geq \alpha$, the function f satisfies $(PS)_c$ and $(epi)_c$;*

(*d*) *we have $|d_{\mathbf{Z}_2}\mathcal{G}_f|(0, \lambda) \neq 0$ whenever $\lambda \geq \alpha$.*

Then there exists a sequence (u_h) of critical points of f with $f(u_h) \to +\infty$.

Proof. Because of assumption (c), the function $\mathcal{G}_f$ satisfies $(PS)_c$ for any $c \geq \alpha$. Then the assertion follows from [36, Theorem (2.7)]. ∎

Now assume that X is a normed space over $\mathbf{R}$ and $f : X \to \overline{\mathbf{R}}$ a function.

Definition 2.14 *For every $u \in X$ with $f(u) \in \mathbf{R}$, $v \in X$ and $\varepsilon > 0$, let $f^\circ_\varepsilon(u; v)$ be the infimum of r's in $\overline{\mathbf{R}}$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{V} : (\mathrm{B}_\delta(u, f(u)) \cap \mathrm{epi}(f)) \times]0, \delta] \to \mathrm{B}_\varepsilon(v)$$

satisfying

$$f(z + t\,\mathcal{V}((z, \mu), t)) \leq \mu + rt$$

whenever $(z, \mu) \in \mathrm{B}_\delta(u, f(u)) \cap \mathrm{epi}(f)$ and $t \in]0, \delta]$. Then let

$$f^\circ(u; v) = \sup_{\varepsilon > 0} f^\circ_\varepsilon(u; v) .$$

Let us recall that the function $f^\circ(u; \cdot)$ is convex, lower semicontinuous and positively homogeneous of degree 1 (see [7, Corollary 4.6]).

Definition 2.15 *For every $u \in X$ with $f(u) \in \mathbf{R}$, we set*

$$\partial f(u) = \{u^* \in X^* : \ \langle u^*, v \rangle \leq f^\circ(u; v) \quad \forall v \in X\} .$$

It turns out that $f^\circ(u; v)$ is greater than or equal to the generalized directional derivative in the sense of Rockafellar (see [14, 47]). Consequently, $\partial f(u)$ contains the subdifferential of f at u in the sense of Clarke. These modified notions of $f^\circ(u; v)$ and $\partial f(u)$ have been introduced in [7, 18], because they are better related with the notion of weak slope and hence more suitable for critical point theory, as the next result shows.

Theorem 2.16 *If $u \in X$ and $f(u) \in \mathbf{R}$, the following facts hold:*

(a) $|df|(u) < +\infty \iff \partial f(u) \neq \emptyset$;

(b) $|df|(u) < +\infty \implies |df|(u) \geq \min\{\|u^*\| : u^* \in \partial f(u)\}$.

Proof. See [7, Theorem 4.13]. ■

However, if $f : X \to \mathbf{R}$ is locally Lipschitz, these notions agree with those of Clarke (see [7, Corollary 4.10]). Thus, in such a case, $f^\circ(u;\cdot)$ is also Lipschitz continuous and we have that

$$\forall u, v \in X : \; f^\circ(u;v) = \limsup_{\substack{z\to u,\, w\to v\\ t\to 0^+}} \frac{f(z+tw)-f(z)}{t}, \tag{2.17}$$

$$\{(u,v) \mapsto f^\circ(u;v)\} \text{ is upper semicontinuous on } X \times X\,. \tag{2.18}$$

3 The general framework

Let $n \geq 1$, $N \geq 1$, Ω be an open subset of $\mathbf{R}^n$ and $1 < p < \infty$. In the following, we wil denote by $\|\cdot\|_q$ the usual norm in L^q $(1 \leq q \leq \infty)$. We now define the functional setting we are interested in.

Let $\mathcal{E} : L^p(\Omega;\mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ be a functional such that:

$(\mathcal{E}_1)$ $\mathcal{E}$ is convex, lower semicontinuous and $0 \in \mathcal{D}(\mathcal{E})$, where

$$\mathcal{D}(\mathcal{E}) = \left\{u \in L^p(\Omega;\mathbf{R}^N) : \mathcal{E}(u) < +\infty\right\};$$

$(\mathcal{E}_2)$ there exists $\vartheta \in C_c(\mathbf{R}^N)$ with $0 \leq \vartheta \leq 1$ and $\vartheta(0) = 1$ such that

$$(\mathcal{E}_2.1) \qquad \forall u \in \mathcal{D}(\mathcal{E}),\ \forall v \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbf{R}^N),\ \forall c > 0:$$
$$\lim_{h\to\infty}\left[\sup_{\substack{\|z-u\|_p\leq c\\ \mathcal{E}(z)\leq c}} \mathcal{E}\left(\vartheta\left(\frac{z}{h}\right)v\right)\right] = \mathcal{E}(v);$$

$$(\mathcal{E}_2.2) \qquad \forall u \in \mathcal{D}(\mathcal{E}) : \lim_{h\to\infty} \mathcal{E}\left(\vartheta\left(\frac{u}{h}\right)u\right) = \mathcal{E}(u).$$

Moreover, let $G : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a function such that

(G_1) $G(\cdot, s)$ is measurable for every $s \in \mathbf{R}^N$;

(G_2) for every $t > 0$ there exists $\alpha_t \in L^1(\Omega)$ such that

$$|G(x,s_1) - G(x,s_2)| \leq \alpha_t(x)|s_1 - s_2|$$

for a.e. $x \in \Omega$ and every $s_1, s_2 \in \mathbf{R}^N$ with $|s_j| \leq t$; for a.e. $x \in \Omega$ we set

$$G^\circ(x,s;\hat{s}) = \gamma^\circ(s;\hat{s}), \qquad \partial_s G(x,s) = \partial\gamma(s),$$

where $\gamma(s) = G(x,s)$;

(G_3) there exist $a_0 \in L^1(\Omega)$ and $b_0 \in \mathbf{R}$ such that

$$G(x,s) \geq -a_0(x) - b_0|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,;$$

(G_4) there exist $a_1 \in L^1(\Omega)$ and $b_1 \in \mathbf{R}$ such that

$$G^\circ(x,s;-s) \leq a_1(x) + b_1|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,.$$

Because of $(\mathcal{E}_1)$ and (G_3), we can define a lower semicontinuous functional $f : L^p(\Omega;\mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ by

$$f(u) = \mathcal{E}(u) + \int_\Omega G(x,u(x))\,dx\,.$$

Remark 3.1 *According to $(\mathcal{E}_1)$, the functional $\mathcal{E}$ is lower semicontinuous. Condition $(\mathcal{E}_2)$ ensures that $\mathcal{E}$ is continuous at least on some particular restrictions.*

Remark 3.2 *If $\{s \mapsto G(x,s)\}$ is of class C^1 for a.e $x \in \Omega$, the estimates in (G_2) and in (G_4) are respectively equivalent to*

$$|s| \leq t \implies |D_s G(x,s)| \leq \alpha_t(x)\,,$$

$$D_s G(x,s) \cdot s \geq -a_1(x) - b_1|s|^p\,.$$

Because of (G_2), for a.e. $x \in \Omega$ and any $t > 0$ and $s \in \mathbf{R}^N$ with $|s| < t$ we have

$$\forall \hat{s} \in \mathbf{R}^N : \; |G^\circ(x,s;\hat{s})| \leq \alpha_t(x)|\hat{s}|\,; \tag{3.3}$$

$$\forall s^* \in \partial_s G(x,s) : \; |s^*| \leq \alpha_t(x)\,. \tag{3.4}$$

In the following, we set $\vartheta_h(s) = \vartheta(s/h)$, where ϑ is a function as in $(\mathcal{E}_2)$, and we fix $M > 0$ such that $\vartheta = 0$ outside $B_M(0)$. Therefore

$$\forall s \in \mathbf{R}^N : \; |s| \geq hM \implies \vartheta_h(s) = 0\,. \tag{3.5}$$

Our first result concerns the connection between the notions of generalized directional derivative and subdifferential in the functional space $L^p(\Omega;\mathbf{R}^N)$ and the more concrete setting of hemivariational inequalities, which also involves the notion of generalized directional derivative, but in $\mathbf{R}^N$.

If $u, v \in L^p(\Omega;\mathbf{R}^N)$, we can define $\int_\Omega G^\circ(x,u;v)\,dx$ if we agree, as in [46], that

$$\int_\Omega G^\circ(x,u;v)\,dx = +\infty \quad \text{whenever} \quad \int_\Omega [G^\circ(x,u;v)]^+\,dx = \int_\Omega [G^\circ(x,u;v)]^-\,dx = +\infty\,.$$

With this convention, $\{v \mapsto \int_\Omega G^\circ(x,u;v)\,dx\}$ is a convex functional from $L^p(\Omega;\mathbf{R}^N)$ into $\overline{\mathbf{R}}$.

Theorem 3.6 *Let $u \in \mathcal{D}(f)$. Then the following facts hold:*

(*a*) *for every $v \in \mathcal{D}(\mathcal{E})$ there exists a sequence (v_h) in $\mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbf{R}^N)$ satisfying $[G^\circ(x,u;v_h - u)]^+ \in L^1(\Omega)$, $\|v_h - v\|_p \to 0$ and $\mathcal{E}(v_h) \to \mathcal{E}(v)$;*

(b) *for every $v \in \mathcal{D}(\mathcal{E})$ we have*

$$f^\circ(u; v-u) \le \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u;v-u)\,dx\,; \tag{3.7}$$

(c) *if $\partial f(u) \neq \emptyset$, we have $G^\circ(x,u;-u) \in L^1(\Omega)$ and*

$$\mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u;v-u)\,dx \ge \int_\Omega u^* \cdot (v-u)\,dx \tag{3.8}$$

for every $u^* \in \partial f(u)$ and $v \in \mathcal{D}(\mathcal{E})$ (the dual space of $L^p(\Omega;\mathbf{R}^N)$ is identified with $L^{p'}(\Omega;\mathbf{R}^N)$ in the usual way);

(d) *if $N=1$, we have $[G^\circ(x,u;v-u)]^+ \in L^1(\Omega)$ for every $v \in L^\infty(\Omega;\mathbf{R}^N)$.*

Proof.
(a) Given $\varepsilon > 0$, by $(\mathcal{E}_2.2)$ we have $\|\vartheta_h(v)v - v\|_p < \varepsilon$ and $|\mathcal{E}(\vartheta_h(v)v) - \mathcal{E}(v)| < \varepsilon$ for h large enough. Then, by $(\mathcal{E}_2.1)$ we get $\|\vartheta_k(u)\vartheta_h(v)v - v\|_p < \varepsilon$ and $|\mathcal{E}(\vartheta_k(u)\vartheta_h(v)v) - \mathcal{E}(v)| < \varepsilon$ for k large enough. Of course $\vartheta_k(u)\vartheta_h(v)v \in L^\infty(\Omega;\mathbf{R}^N)$ and by (3.3) we have

$$\begin{aligned} G^\circ(x,u;\vartheta_k(u)\vartheta_h(v)v - u) &\le \vartheta_k(u)\vartheta_h(v)G^\circ(x,u;v-u) + \\ &\quad +(1-\vartheta_k(u)\vartheta_h(v))G^\circ(x,u;-u) \le \\ &\le (h+k)M\alpha_{kM}(x) + [G^\circ(x,u;-u)]^+\,. \end{aligned}$$

From (G_4) we infer that $[G^\circ(x,u;-u)]^+ \in L^1(\Omega)$ and assertion (a) follows.
(b) Without loss of generality, we may assume that $[G^\circ(x,u;v-u)]^+ \in L^1(\Omega)$. Suppose first that $v \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbf{R}^N)$ and take $\varepsilon > 0$.

We claim that for every $z \in L^p(\Omega;\mathbf{R}^N)$, $t \in]0,1/2]$ and $h \ge 1$ with $hM > \|v\|_\infty$, we have

$$\frac{G(x, z + t(\vartheta_h(z)v - z)) - G(x,z)}{t} \le 2\left(\|v\|_\infty\, \alpha_{hM} + a_1 + b_1(|z|+|v|)^p\right)\,. \tag{3.9}$$

In fact, for a.e. $x \in \Omega$, by Lebourg's Theorem (see e.g. [14]) there exist $\overline{t} \in]0,t[$ and $u^* \in \partial_s G(x, z + \overline{t}(\vartheta_h(z)v - z))$ such that

$$\begin{aligned} \frac{G(x, z + t(\vartheta_h(z)v - z)) - G(x,z)}{t} &= u^* \cdot (\vartheta_h(z)v - z) = \\ &= \frac{1}{1-\overline{t}}\left[\vartheta_h(z)u^* \cdot v - u^* \cdot (z + \overline{t}(\vartheta_h(z)v - z))\right]\,. \end{aligned}$$

By (3.4) and (3.5), it easily follows that

$$\frac{|\vartheta_h(z)u^* \cdot v|}{1-\overline{t}} \le 2\,\|v\|_\infty\, \alpha_{hM}\,.$$

On the other hand, from (G_4) we deduce that for a.e. $x \in \Omega$

$$\begin{aligned} \frac{u^* \cdot (z + \overline{t}(\vartheta_h(z)v - z))}{1-\overline{t}} &\ge -\frac{1}{1-\overline{t}} G^\circ(x, z + \overline{t}(\vartheta_h(z)v - z); -(z + \overline{t}(\vartheta_h(z)v - z)) \ge \\ &\ge -\frac{1}{1-\overline{t}}(a_1 + b_1|z + \overline{t}(\vartheta_h(z)v - z)|^p) \ge -2\left(a_1 + b_1(|z|+|v|)^p\right)\,. \end{aligned}$$

Then (3.9) easily follows.

For a.e. $x \in \Omega$ we have

$$\begin{aligned} G^\circ(x,u;\vartheta_h(u)v-u) &\leq \vartheta_h(u)G^\circ(x,u;v-u)+(1-\vartheta_h(u))G^\circ(x,u;-u) \leq \\ &\leq [G^\circ(x,u;v-u)]^+ + [G^\circ(x,u;-u)]^+ . \end{aligned}$$

Furthermore, for a.e. $x \in \Omega$ and every $s \in \mathbf{R}^N$, (G_2) implies $G^\circ(x,s;\cdot)$ to be Lipschitz continuous, so in particular

$$\lim_h G^\circ(x,u;\vartheta_h(u)v-u) = G^\circ(x,u;v-u) \qquad \text{a.e. in } \Omega .$$

Then, given

$$\lambda > \int_\Omega G^\circ(x,u;v-u)\,dx ,$$

by Fatou's Lemma there exists $\overline{h} \geq 1$ such that

$$\forall h \geq \overline{h} : \int_\Omega G^\circ(x,u;\vartheta_h(u)v-u)\,dx < \lambda \quad \text{and} \quad \|\vartheta_h(u)v - v\|_p < \varepsilon . \tag{3.10}$$

By the lower semicontinuity of $\mathcal{G}$, there exists $\overline{\delta} \in]0,1/2]$ such that for every $z \in \mathrm{B}_{\overline{\delta}}(u)$ it is $\mathcal{G}(z) \geq \mathcal{G}(u) - \frac{1}{2}$. Then for every $(z,\mu) \in \mathrm{B}_{\overline{\delta}}(u,f(u)) \cap \mathrm{epi}\,(f)$ it follows

$$\mathcal{E}(z) \leq \mu - \mathcal{G}(z) \leq \mu + \frac{1}{2} - \mathcal{G}(u) \leq f(u) + \overline{\delta} - \mathcal{G}(u) + \frac{1}{2} \leq \mathcal{E}(u) + 1 .$$

Let now $\sigma > 0$. By assumptions $(\mathcal{E}_1)$ and $(\mathcal{E}_2.1)$ there exist $h \geq \overline{h}$ and $\delta \leq \overline{\delta}$ such that

$$\|v\|_\infty < hM ,$$

$$\mathcal{E}(z) > \mathcal{E}(u) - \sigma , \quad \mathcal{E}(\vartheta_h(z)v) < \mathcal{E}(v) + \sigma , \quad \|(\vartheta_h(z)v - z) - (v-u)\|_p < \varepsilon ,$$

for any $z \in \mathrm{B}_\delta(u)$ with $\mathcal{E}(z) \leq \mathcal{E}(u) + 1$.

Taking into account (2.17), (3.9) and (3.10), we deduce by Fatou's Lemma that, possibly reducing δ, for any $t \in]0,\delta]$ and for any $z \in \mathrm{B}_\delta(u)$ we have

$$\int_\Omega \frac{G(x,z+t(\vartheta_h(z)v-z)) - G(x,z)}{t}\,dx < \lambda .$$

Now let $\mathcal{V} : (\mathrm{B}_\delta(u,f(u)) \cap \mathrm{epi}\,(f)) \times]0,\delta] \to \mathrm{B}_\varepsilon(v-u)$ be defined setting

$$\mathcal{V}((z,\mu),t) = \vartheta_h(z)v - z .$$

Since $\mathcal{V}$ is evidently continuous and

$$\begin{aligned} f(z+t\mathcal{V}((z,\mu),t)) &= f(z+t(\vartheta_h(z)v-z)) \leq \\ &\leq \mathcal{E}(z) + t\,(\mathcal{E}(\vartheta_h(z)v) - \mathcal{E}(z)) + \mathcal{G}(z+t(\vartheta_h(z)v-z) \leq \\ &\leq \mathcal{E}(z) + (\mathcal{E}(v) - \mathcal{E}(u) + 2\sigma)t + \mathcal{G}(z) + \lambda t = \\ &= f(z) + (\mathcal{E}(v) - \mathcal{E}(u) + \lambda + 2\sigma)t , \end{aligned}$$

we have

$$f_\varepsilon^\circ(u; v-u) \leq \mathcal{E}(v) - \mathcal{E}(u) + \lambda + 2\sigma\,.$$

By the arbitrariness of $\sigma > 0$ and $\lambda > \int_\Omega G^\circ(x,u; v-u)\,dx$, it follows

$$f_\varepsilon^\circ(u; v-u) \leq \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u; v-u)\,dx\,.$$

Passing to the limit as $\varepsilon \to 0^+$, we get (3.7) when $v \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega; \mathbf{R}^N)$.

Let us now treat the general case. If we set $v_h = \vartheta_h(v)v$, we have $v_h \in L^\infty(\Omega; \mathbf{R}^N)$. Arguing as before, it is easy to see that

$$G^\circ(x,u; v_h - u) \leq [G^\circ(x,u; v-u)]^+ + [G^\circ(x,u; -u)]^+\,,$$

so that

$$\limsup_h \int_\Omega G^\circ(x,u; v_h - u)\,dx \leq \int_\Omega G^\circ(x,u; v-u)\,dx\,.$$

On the other hand, by the previous step it holds

$$f^\circ(u; v_h - u) \leq \mathcal{E}(v_h) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u; v_h - u)\,dx\,.$$

Passing to the lower limit as $h \to \infty$ and taking into account the lower semicontinuity of $f^\circ(u,\cdot)$ and $(\mathcal{E}_2.2)$, we get (3.7).

(c) We already know that $[G^\circ(x,u;-u)]^+ \in L^1(\Omega)$. If we choose $v = 0$ in (3.7), we obtain

$$f^\circ(u; -u) \leq \mathcal{E}(0) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u; -u)\,dx\,.$$

Since $\partial f(u) \neq \emptyset$, it is $f^\circ(u;-u) > -\infty$, hence

$$\int_\Omega [G^\circ(x,u;-u)]^-\,dx < +\infty\,.$$

Finally, if $u^* \in \partial f(u)$ we have by definition that

$$f^\circ(u; v-u) \geq \int_\Omega u^* \cdot (v-u)\,dx$$

and (3.8) follows from (3.7).

(d) From (3.3) it readily follows that $G^\circ(x,u; v-u)$ is summable where $|u(x)| \leq \|v\|_\infty$. On the other hand, where $|u(x)| > \|v\|_\infty$ we have

$$G^\circ(x,u; v-u) = \left(1 - \frac{v}{u}\right) G^\circ(x,u; -u)$$

and the assertion follows from (G_4). ■

Since f is only lower semicontinuous, we are interested in the verification of the condition $(epi)_c$. For this purpose, we consider an assumption (G_3') on G stronger than (G_3).

Theorem 3.11 *Assume that*

(G_3') *there exist* $a \in L^1(\Omega)$ *and* $b \in \mathbf{R}$ *such that*

$$|G(x,s)| \leq a(x) + b|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N .$$

Then for every $(u,\lambda) \in \operatorname{epi}(f)$ *with* $\lambda > f(u)$ *it is* $|d\mathcal{G}_f|(u,\lambda) = 1$. *Moreover, if* $\mathcal{E}$ *and* $G(x,\cdot)$ *are even, for every* $\lambda > f(0)$ *we have* $|d_{\mathbf{Z}_2}\mathcal{G}_f|(0,\lambda) = 1$.

Proof. Let $\varrho > 0$. Since

$$\forall \tau \in [0,1]: \; G^\circ(x,u;\tau u - u) = (1-\tau)G^\circ(x,u;-u) \leq [G^\circ(x,u;-u)]^+ ,$$

by $(\mathcal{E}_2.2)$ and (G_4) there exists $\overline{h} \geq 1$ such that

$$\|\vartheta_{\overline{h}}(u)u - u\|_p < \varrho , \qquad \mathcal{E}(\vartheta_{\overline{h}}(u)u) < \mathcal{E}(u) + \varrho ,$$

$$\forall h \geq \overline{h}: \; \int_\Omega G^\circ(x,u;\vartheta_h(u)\vartheta_{\overline{h}}(u)u - u)\,dx < \varrho .$$

Set $v = \vartheta_{\overline{h}}(u)u$.

By $(\mathcal{E}_2.1)$ there exist $h \geq \overline{h}$ and $\delta \in]0,1]$ such that

$$\|\vartheta_h(z)v - z\|_p < \varrho , \qquad \mathcal{E}(\vartheta_h(z)v) < \mathcal{E}(u) + \varrho ,$$

whenever $\|z-u\|_p < \delta$ and $\mathcal{E}(z) \leq \lambda + 1 - \mathcal{G}(u) + \varrho$.

By decreasing δ, from (G_3'), (3.9) and (2.17) we deduce that

$$|\mathcal{G}(z) - \mathcal{G}(u)| < \varrho , \qquad \int_\Omega \frac{G(x,z+t(\vartheta_h(z)v - z)) - G(x,z)}{t}\,dx < \varrho$$

whenever $\|z-u\|_p < \delta$ and $0 < t \leq \delta$.

Define a continuous map

$$\mathcal{H} : \{z \in \mathrm{B}_\delta(u) : f(z) < \lambda + \delta\} \times [0,\delta] \to X$$

by $\mathcal{H}(z,t) = z + t(\vartheta_h(z)v - z)$. It is readily seen that $\|\mathcal{H}(z,t) - z\|_p \leq \varrho t$.

If $z \in \mathrm{B}_\delta(u)$, $f(z) < \lambda + \delta$ and $0 \leq t \leq \delta$, we have

$$\mathcal{E}(z) = f(z) - \mathcal{G}(z) < \lambda + \delta - \mathcal{G}(u) + \varrho \leq \lambda + 1 - \mathcal{G}(u) + \varrho ,$$

hence, taking into account the convexity of $\mathcal{E}$,

$$\mathcal{E}(z + t(\vartheta_h(z)v - z)) \leq \mathcal{E}(z) + t(\mathcal{E}(\vartheta_h(z)v) - \mathcal{E}(z)) \leq \mathcal{E}(z) + t(\mathcal{E}(u) - \mathcal{E}(z) + \varrho) .$$

Moreover, we also have

$$\mathcal{G}(z + t(\vartheta_h(z)v - z)) \leq \mathcal{G}(z) + t\varrho \leq \mathcal{G}(z) + t(\mathcal{G}(u) - \mathcal{G}(z) + 2\varrho) .$$

Therefore

$$f(z + t(\vartheta_h(z)v - z)) \leq f(z) + t(f(u) - f(z) + 3\varrho) .$$

and the first assertion follows by Corollary 2.11.

Now assume that $\mathcal{E}$ and $G(x,\cdot)$ are even and that $u=0$. Then, in the previous argument, we have $v=0$, so that $\mathcal{H}(-z,t)=-\mathcal{H}(z,t)$ and the second assertion also follows. ■

Now we want to provide a criterion which helps in the verification of the Palais-Smale condition. For this purpose, we consider further assumptions on $\mathcal{E}$, which ensure a suitable coerciveness, and a new condition (G_4') on G, stronger than (G_4), which is a kind of one-sided subcritical growth condition.

Theorem 3.12 *Let $c \in \mathbf{R}$. Assume that*

$(\mathcal{E}_3)$ *for every (u_h) bounded in $L^p(\Omega;\mathbf{R}^N)$ with $(\mathcal{E}(u_h))$ bounded, there exists a subsequence (u_{h_k}) and a function $u \in L^p(\Omega;\mathbf{R}^N)$ such that*

$$\lim_{k\to\infty} u_{h_k}(x) = u(x) \qquad \text{for a.e. } x \in \Omega\,;$$

$(\mathcal{E}_4)$ *if (u_h) is a sequence in $L^p(\Omega;\mathbf{R}^N)$ weakly convergent to $u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(u_h)$ converges to $\mathcal{E}(u)$, then (u_h) converges to u strongly in $L^p(\Omega;\mathbf{R}^N)$;*

(G_4') *for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that*

$$G^\circ(x,s;-s) \le a_\varepsilon(x) + \varepsilon|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,.$$

Then any $(PS)_c$-sequence (u_h) for f bounded in $L^p(\Omega;\mathbf{R}^N)$ admits a subsequence strongly convergent in $L^p(\Omega;\mathbf{R}^N)$.

Proof. From (G_3) we deduce that $(\mathcal{G}(u_h))$ is bounded from below. Taking into account $(\mathcal{E}_1)$, it follows that $(\mathcal{E}(u_h))$ is bounded. By $(\mathcal{E}_3)$ there exists a subsequence, still denoted by (u_h), converging weakly in $L^p(\Omega;\mathbf{R}^N)$ and a.e. to some $u \in \mathcal{D}(\mathcal{E})$.

Given $\varepsilon > 0$, by $(\mathcal{E}_2.2)$ and (G_4) we may find $k_0 \ge 1$ such that

$$\mathcal{E}(\vartheta_{k_0}(u)u) < \mathcal{E}(u) + \varepsilon\,,$$

$$\int_\Omega (1-\vartheta_{k_0}(u))G^\circ(x,u;-u)\,dx < \varepsilon\,.$$

Since $\vartheta_{k_0}(u)u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbf{R}^N)$, by $(\mathcal{E}_2.1)$ there exists $k_1 \ge k_0$ such that

$$\forall h \in \mathbf{N}: \quad \mathcal{E}(\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u) < \mathcal{E}(u) + \varepsilon\,, \tag{3.13}$$

$$\int_\Omega (1-\vartheta_{k_1}(u)\vartheta_{k_0}(u))G^\circ(x,u;-u)\,dx < \varepsilon\,.$$

It follows that $\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u \in \mathcal{D}(\mathcal{E})$. Moreover, from (3.3) and (G_4') we get

$$\begin{aligned} &G^\circ(x,u_h;\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u - u_h) \le \\ &\le \vartheta_{k_1}(u_h)G^\circ(x,u_h;\vartheta_{k_0}(u)u-u_h) + (1-\vartheta_{k_1}(u_h))G^\circ(x,u_h;-u_h) \le \\ &\le \alpha_{k_1 M}(x)(k_0M + k_1M) + a_\varepsilon(x) + \varepsilon|u_h|^p\,. \end{aligned}$$

From (2.18) and Fatou's Lemma we deduce that

$$\limsup_{h\to\infty}\int_\Omega [G^\circ(x,u_h;\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h)-\varepsilon|u_h|^p]\,dx \le$$
$$\le \int_\Omega [G^\circ(x,u;\vartheta_{k_1}(u)\vartheta_{k_0}(u)u-u)-\varepsilon|u|^p]\,dx \le$$
$$\le \int_\Omega (1-\vartheta_{k_1}(u)\vartheta_{k_0}(u))G^\circ(x,u;-u)\,dx < \varepsilon\,,$$

hence

$$\limsup_{h\to\infty}\int_\Omega G^\circ(x,u_h;\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h)\,dx < \varepsilon\sup_h\|u_h\|_p^p+\varepsilon\,. \tag{3.14}$$

Since (u_h) is a $(PS)_c$-sequence, by Theorem 2.16 there exists $u_h^*\in\partial f(u_h)$ with $\|u_h^*\|_{p'}\le |df|\,(u_h)$, so that $\lim\limits_{h\to\infty}\|u_h^*\|_{p'}=0$. Applying (c) of Theorem 3.6, we get

$$\mathcal{E}(\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u)\ge\mathcal{E}(u_h)-\int_\Omega G^\circ(x,u_h;\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h)\,dx+$$
$$+\int_\Omega u_h^*\cdot(\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u-u_h)\,dx\,.$$

Taking into account (3.13), (3.14) and passing to the upper limit, we obtain

$$\limsup_{h\to\infty}\mathcal{E}(u_h)\le\mathcal{E}(u)+2\varepsilon+\varepsilon\sup_h\|u_h\|_p^p\,.$$

By the arbitrariness of $\varepsilon>0$, we finally have

$$\limsup_{h\to\infty}\mathcal{E}(u_h)\le\mathcal{E}(u)$$

and the strong convergence of (u_h) to u follows from $(\mathcal{E}_4)$. ■

4 Area type functionals

Let $n\ge 2$, $N\ge 1$, Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and let

$$\Psi:\mathbf{R}^{nN}\to\mathbf{R}$$

be a convex function satisfying

$$(\Psi)\qquad \begin{cases}\Psi(0)=0,\ \Psi(\xi)>0 \text{ for any } \xi\ne 0 \text{ and}\\ \text{there exists } c>0 \text{ such that } \Psi(\xi)\le c|\xi| \text{ for any } \xi\in\mathbf{R}^{nN}.\end{cases}$$

We want to study the functional $\mathcal{E}:L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)\to\mathbf{R}\cup\{+\infty\}$ defined by

$$\mathcal{E}(u)=\begin{cases}\displaystyle\int_\Omega\Psi(Du^a)\,dx+\int_\Omega\Psi^\infty\left(\frac{Du^s}{|Du^s|}\right)d|Du^s|(x)+\\ \displaystyle+\int_{\partial\Omega}\Psi^\infty(u\otimes\nu)\,d\mathcal{H}^{n-1}(x) & \text{if } u\in BV(\Omega;\mathbf{R}^N),\\ +\infty & \text{if } u\in L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)\setminus BV(\Omega;\mathbf{R}^N),\end{cases}$$

where $Du = Du^a\,dx + Du^s$ is the Lebesgue decomposition of Du, $|Du^s|$ is the total variation of Du^s, $Du^s/|Du^s|$ is the Radon-Nikodym derivative of Du^s with respect to $|Du^s|$, Ψ^∞ is the recession functional associated with Ψ, ν is the outer normal to Ω and the trace of u on $\partial\Omega$ is still denoted by u (see e.g. [4, 29]).

Theorem 4.1 *The functional $\mathcal{E}$ satisfies conditions $(\mathcal{E}_1)$, $(\mathcal{E}_2)$, $(\mathcal{E}_3)$ and $(\mathcal{E}_4)$.*

The section will be devoted to the proof of this result. We begin establishing some technical lemmas. For notions concerning the space BV, such as those of $\tilde{u}$, S_u, u^+ and u^-, we refer the reader to [2, 3].

In $BV(\Omega;\mathbf{R}^N)$ we will consider the norm

$$\|u\|_{BV} = \int_\Omega |Du^a|\,dx + |Du^s|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{n-1}(x)\,,$$

which is equivalent to the standard norm of $BV(\Omega;\mathbf{R}^N)$.

Lemma 4.2 *For every $u \in BV(\Omega;\mathbf{R}^N)$ and every $\varepsilon > 0$ there exists $v \in C_c^\infty(\Omega;\mathbf{R}^N)$ such that*

$$\|v-u\|_{\frac{n}{n-1}} < \varepsilon\,, \qquad \left|\int_\Omega |Dv|\,dx - \|u\|_{BV}\right| < \varepsilon\,, \qquad |\mathcal{E}(v) - \mathcal{E}(u)| < \varepsilon\,, \qquad \|v\|_\infty \leq \operatorname*{ess\,sup}_\Omega |u|\,.$$

Proof. Let $\delta > 0$, let $R > 0$ with $\overline{\Omega} \subseteq \mathrm{B}_R(0)$ and let

$$\vartheta_h(x) = 1 - \min\left\{\max\left\{\frac{h+1}{h}[1 - h\,d(x,\mathbf{R}^n\setminus\Omega)], 0\right\}, 1\right\}.$$

Define $\hat{u} \in BV(\mathrm{B}_R(0);\mathbf{R}^N)$ by

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega\,, \\ 0 & \text{if } x \in \mathrm{B}_R(0)\setminus\Omega\,. \end{cases}$$

According to [11, Lemma 7.4 and formula (7.2)], if h is sufficiently large, we have that $\vartheta_h u \in BV(\Omega;\mathbf{R}^N)$, $\|\vartheta_h u - u\|_{\frac{n}{n-1}} < \delta$ and

$$\begin{aligned} &\int_\Omega \sqrt{1 + |D(\vartheta_h u)^a|^2}\,d\mathcal{L}^n + |D(\vartheta_h u)^s|(\Omega) < \\ &< \int_\Omega \sqrt{1+|Du^a|^2}\,d\mathcal{L}^n + |Du^s|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{n-1} + \delta = \\ &= \int_{\mathrm{B}_R(0)} \sqrt{1+|D\hat{u}^a|^2}\,d\mathcal{L}^n + |D\hat{u}^s|(\mathrm{B}_R(0)) + \delta\,. \end{aligned}$$

Moreover, $\vartheta_h u$ has compact support in Ω and $\operatorname*{ess\,sup}_\Omega |\vartheta_h u| \leq \operatorname*{ess\,sup}_\Omega |u|$.

If we regularize $\vartheta_h u$ by convolution, we easily get $v \in C_c^\infty(\Omega;\mathbf{R}^N)$ with

$$\|v\|_\infty \leq \operatorname*{ess\,sup}_\Omega |u|\,, \qquad \|v-u\|_{\frac{n}{n-1}} < \delta$$

and

$$\int_\Omega \sqrt{1+|Dv|^2}\,d\mathcal{L}^n < \int_{\mathrm{B}_R(0)} \sqrt{1+|D\hat{u}^a|^2}\,d\mathcal{L}^n + |D\hat{u}^s|(\mathrm{B}_R\,(0)) + \delta\,.$$

Since

$$\|u\|_{BV} = \int_{\mathrm{B}_R(0)} |D\hat{u}^a|\,dx + |D\hat{u}^s|(\mathrm{B}_R\,(0))\,,$$

$$\mathcal{E}(u) = \int_{\mathrm{B}_R(0)} \Psi(D\hat{u}^a)\,dx + \int_{\mathrm{B}_R(0)} \Psi^\infty\left(\frac{D\hat{u}^s}{|D\hat{u}^s|}\right)\,d|D\hat{u}^s|\,,$$

by the results of [45] the assertion follows (see also [4, Fact 3.1]). ∎

Lemma 4.3 *The following facts hold:*

(*a*) $\Psi : \mathbf{R}^{nN} \to \mathbf{R}$ *is Lipschitz continuous of some constant* $\mathrm{Lip}(\Psi) > 0$*;*

(*b*) *for any* $\xi \in \mathbf{R}^{nN}$ *and* $s \in [0,1]$ *we have* $\Psi(s\xi) \le s\Psi(\xi)$*;*

(*c*) *for every* $\sigma > 0$ *there exists* $d_\sigma > 0$ *such that*

$$\forall \xi \in \mathbf{R}^{nN}: \quad \Psi(\xi) \ge d_\sigma(|\xi| - \sigma)\,;$$

(*d*) $\mathcal{E} : BV(\Omega;\mathbf{R}^N) \to \mathbf{R}$ *is Lipschitz continuous of constant* $\mathrm{Lip}(\Psi)$*;*

(*e*) *if* σ *and* d_σ *are as in* (*c*)*, we have*

$$\forall u \in BV(\Omega;\mathbf{R}^N): \quad \mathcal{E}(u) \ge d_\sigma\Big(\|u\|_{BV} - \sigma\mathcal{L}^n(\Omega)\Big)\,.$$

Proof. Properties (*a*) and (*b*) easily follow from the convexity of Ψ and assumption (Ψ).

To prove (*c*), assume by contradiction that $\sigma > 0$ and (ξ_h) is a sequence with $\Psi(\xi_h) < \frac{1}{h}(|\xi_h| - \sigma)$. If $|\xi_h| \to +\infty$, we have eventually

$$\Psi\left(\frac{\xi_h}{|\xi_h|}\right) \le \frac{\Psi(\xi_h)}{|\xi_h|} < \frac{1}{h}\left(1 - \frac{\sigma}{|\xi_h|}\right)\,.$$

Up to a subsequence, $(\xi_h/|\xi_h|)$ is convergent to some $\eta \neq 0$ with $\Psi(\eta) \le 0$, which is impossible. Since $|\xi_h|$ is bounded, up to a subsequence we have $\xi_h \to \xi$ with $|\xi| \ge \sigma$ and $\Psi(\xi) \le 0$, which is again impossible.

Finally, (*d*) easily follows from (*a*) and the definition of $\|\cdot\|_{BV}$, while (*e*) follows from (*c*) (see e.g. [37, Lemma 4.1]). ∎

Let now $\vartheta \in C^1_c(\mathbf{R}^N)$ with $0 \le \vartheta \le 1$, $\|\nabla\vartheta\|_\infty \le 2$, $\vartheta(s) = 1$ for $|s| \le 1$ and $\vartheta(s) = 0$ for $|s| \ge 2$. Define $\vartheta_h : \mathbf{R}^N \to \mathbf{R}$ and $T_h, R_h : \mathbf{R}^N \to \mathbf{R}^N$ by

$$\vartheta_h(s) = \vartheta\left(\frac{s}{h}\right)\,, \quad T_h(s) = \vartheta_h(s)s\,, \quad R_h(s) = (1 - \vartheta_h(s))s\,.$$

Lemma 4.4 *There exists a constant $c_\Psi > 0$ such that*

$$\mathcal{E}\left(\vartheta\left(\frac{u}{h}\right)v\right) \le \mathcal{E}(v) + \frac{c_\Psi}{h}\|v\|_\infty \|u\|_{BV},$$

$$\mathcal{E}(T_h \circ u) \le \mathcal{E}(u) + c_\Psi \Bigg[|Du|(\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}) +$$

$$+ \int_{\{x \in S_u : |u^+(x)| > h \text{ or } |u^-(x)| > h\}} |u^+ - u^-| \, d\mathcal{H}^{n-1}(x) + \int_{\{x \in \partial\Omega : |u(x)| > h\}} |u| \, d\mathcal{H}^{n-1}(x) \Bigg],$$

$$\mathcal{E}(T_h \circ w) + \mathcal{E}(R_h \circ w) \le \mathcal{E}(w) + c_\Psi \int_{\{x \in \Omega : h < |w(x)| < 2h\}} |Dw| \, dx$$

whenever $h \ge 1$, $u \in BV(\Omega; \mathbf{R}^N)$, $v \in BV(\Omega; \mathbf{R}^N) \cap L^\infty(\Omega; \mathbf{R}^N)$ and $w \in C_c^\infty(\Omega; \mathbf{R}^N)$.

Proof. Suppose first that $u, v \in C_c^\infty(\Omega; \mathbf{R}^N)$. Then, since

$$D\left[\vartheta\left(\frac{u}{h}\right)v\right] = \vartheta\left(\frac{u}{h}\right)Dv + \frac{1}{h} v \otimes \left[D\vartheta\left(\frac{u}{h}\right)Du\right],$$

by (Ψ) and Lemma 4.3 it follows that

$$\mathcal{E}\left(\vartheta\left(\frac{u}{h}\right)v\right) \le \mathcal{E}(v) + \mathrm{Lip}(\Psi)\frac{\|D\vartheta\|_\infty}{h}\|v\|_\infty \int_\Omega |Du| \, dx. \tag{4.5}$$

In the general case, let us consider two sequences (u_k), (v_k) in $C_c^\infty(\Omega; \mathbf{R}^N)$ converging to u, v in $L^1(\Omega; \mathbf{R}^N)$ with $\int_\Omega |Du_k| \, dx \to \|u\|_{BV}$, $\mathcal{E}(v_k) \to \mathcal{E}(v)$ and $\|v_k\|_\infty \le \|v\|_\infty$. Passing to the lower limit in (4.5), we obtain the first inequality in the assertion.

To prove the second inequality, we first observe that by Lemma 4.3 we have

$$\mathcal{E}(T_h \circ u) \le \mathcal{E}(u) + \mathrm{Lip}(\Psi)\|R_h \circ u\|_{BV}. \tag{4.6}$$

In order to estimate the last term in (4.6), we apply the chain rule of [2, 49]. Since $R_h(s) = 0$ if $|s| \le h$ and $\|DR_h\|_\infty \le k_\vartheta$ for some $k_\vartheta > 0$, we have

$$\int_\Omega |D(R_h(u))^a| \, dx \le \int_{\Omega \setminus S_u} |DR_h(\tilde{u})||Du^a| \, dx \le k_\vartheta \int_{\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}} |Du^a| \, dx,$$

$$\left|D(R_h(u))^s\right|(\Omega) \le \int_{\Omega \setminus S_u} |DR_h(\tilde{u})| \, d|Du^s|(x) + \int_{S_u} |R_h(u^+) - R_h(u^-)| \, d\mathcal{H}^{n-1}(x) \le$$

$$\le k_\vartheta \Big(|Du^s| \left(\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}\right) + \int_{\{x \in S_u : |u^+(x)| > h \text{ or } |u^-(x)| > h\}} |u^+ - u^-| \, d\mathcal{H}^{m-1}(x) \Big)$$

and

$$\int_{\partial\Omega} |R_h(u)| \, d\mathcal{H}^{n-1}(x) \le k_\vartheta \int_{\{x \in \partial\Omega : |u(x)| > h\}} |u| \, d\mathcal{H}^{n-1}(x).$$

Combining these three estimates, we get

$$\text{(4.7)}\ \|R_h \circ u\|_{BV} \le k_\vartheta \Bigg(\int_{\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}} |Du^a| \, dx + |Du^s|(\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}) +$$

$$+\int_{\{x\in S_u:|u^+(x)|>h \text{ or } |u^-(x)|>h\}}|u^+-u^-|\,d\mathcal{H}^{m-1}(x)+\int_{\{x\in\partial\Omega:|u(x)|>h\}}|u|\,d\mathcal{H}^{n-1}(x)\Big).$$

Then the second inequality follows from (4.6) and (4.7).

Again, since Ψ is Lipschitz continuous, we have

$$\begin{aligned}\left|\int_\Omega \Psi(D(T_h\circ w))\,dx-\int_\Omega\Psi(\vartheta_h(w)Dw)\,dx\right| &\le \frac{\mathrm{Lip}(\Psi)}{h}\int_\Omega\left|D\vartheta\left(\frac{w}{h}\right)Dw\right||w|\,dx\le\\ &\le 2\,\mathrm{Lip}(\Psi)\|\nabla\vartheta\|_\infty\int_{\{h<|w|<2h\}}|Dw|\,dx\,.\end{aligned}$$

In a similar way, it is also

$$\left|\int_\Omega\Psi(D(R_h\circ w))\,dx-\int_\Omega\Psi((1-\vartheta_h(w))Dw)\,dx\right|\le 2\,\mathrm{Lip}(\Psi)\|\nabla\vartheta\|_\infty\int_{\{h<|w|<2h\}}|Dw|\,dx\,.$$

Hence, combining the last two estimates and taking into account (b) of Lemma 4.3, we get

$$\begin{aligned}&\int_\Omega\Psi(D(T_h\circ w))\,dx+\int_\Omega\Psi(D(R_h\circ w))\,dx\le\\ &\le\int_\Omega\Psi(Dw)\,dx+4\,\mathrm{Lip}(\Psi)\|\nabla\vartheta\|_\infty\int_{\{h<|w|<2h\}}|Dw|\,dx\end{aligned}$$

and the proof is complete. ■

Lemma 4.8 *Let (u_h) be a sequence in $C_c^\infty(\Omega;\mathbf{R}^N)$ and assume that (u_h) is bounded in $BV(\Omega;\mathbf{R}^N)$.*

Then for every $\varepsilon>0$ and every $\overline{k}\in\mathbf{N}$ there exists $k\ge\overline{k}$ such that

$$\liminf_{h\to\infty}\int\limits_{\{k<|u_h|<2k\}}|Du_h|\,dx<\varepsilon\,.$$

Proof. Let $m\ge1$ be such that

$$\sup_h\int_\Omega|Du_h|\,dx\le\frac{m\varepsilon}{2}$$

and let $i_0\in\mathbf{N}$ with $2^{i_0}\ge\overline{k}$. Then, since

$$\sum_{i=i_0}^{i_0+m-1}\int_{\{2^i<|u_h|<2^{i+1}\}}|Du_h|\,dx\le\int_\Omega|Du_h|\,dx\le\frac{m\varepsilon}{2}\,,$$

there exists i_h between i_0 and i_0+m-1 such that

$$\int_{\{2^{i_h}<|u_h|<2^{i_h+1}\}}|Du_h|\,dx\le\frac{\varepsilon}{2}\,.$$

Passing to a subsequence (i_{h_j}), we can suppose $i_{h_j}\equiv i\ge i_0$, and setting $k=2^i$ we get

$$\forall j\in\mathbf{N}:\int\limits_{\{k<|u_{h_j}|<2k\}}|Du_{h_j}|\,dx\le\frac{\varepsilon}{2}\,.$$

Then the assertion follows. ■

Lemma 4.9 *Let (u_h) be a sequence in $C_c^\infty(\Omega;\mathbf{R}^N)$ and let $u \in BV(\Omega;\mathbf{R}^N)$ with $\|u_h - u\|_1 \to 0$ and $\mathcal{E}(u_h) \to \mathcal{E}(u)$.*

Then for every $\varepsilon > 0$ and every $\overline{k} \in \mathbf{N}$ there exists $k \geq \overline{k}$ such that

$$\liminf_{h\to\infty} \|R_k \circ u_h\|_{BV} < \varepsilon\,.$$

Proof. Given $\varepsilon > 0$, let $d > 0$ be such that

$$\forall \xi \in \mathbf{R}^{nN}: \quad \Psi(\xi) \geq d\left(|\xi| - \frac{\varepsilon}{3\mathcal{L}^n(\Omega)}\right),$$

according to Lemma 4.3. Let also $c_\Psi > 0$ be as in Lemma 4.4. By (4.7) and Lemma 4.8, there exists $k \geq \overline{k}$ such that

$$\|R_k \circ u\|_{BV} < \frac{d\varepsilon}{3\mathrm{Lip}(\Psi)}\,,$$

$$\liminf_{h\to\infty} \int_{\{k<|u_h|<2k\}} |Du_h|\,dx < \frac{d\varepsilon}{3c_\Psi}\,.$$

From Lemma 4.4 we deduce that

$$\begin{aligned}
\mathcal{E}(T_k \circ u) + \liminf_{h\to\infty} \mathcal{E}(R_k \circ u_h) \;&\leq\; \liminf_{h\to\infty} \mathcal{E}(T_k \circ u_h) + \liminf_{h\to\infty} \mathcal{E}(R_k \circ u_h) \leq \\
&\leq\; \liminf_{h\to\infty}\Big(\mathcal{E}(T_k \circ u_h) + \mathcal{E}(R_k \circ u_h)\Big) \leq \\
&\leq\; \mathcal{E}(u) + c_\Psi \liminf_{h\to\infty} \int_{\{k<|u_h|<2k\}} |Du_h|\,dx < \\
&<\; \mathcal{E}(u) + \frac{d\varepsilon}{3} \leq \mathcal{E}(T_k \circ u) + \mathrm{Lip}(\Psi)\|R_k \circ u\|_{BV} + \frac{d\varepsilon}{3} < \\
&<\; \mathcal{E}(T_k \circ u) + \frac{2}{3}d\varepsilon\,,
\end{aligned}$$

whence

$$\liminf_{h\to\infty} \mathcal{E}(R_k \circ u_h) < \frac{2}{3}d\varepsilon\,.$$

On the other hand, by Lemma 4.3 we have

$$\mathcal{E}(R_k \circ u_h) \geq d\left(\|R_k \circ u_h\|_{BV} - \frac{\varepsilon}{3}\right)$$

and the assertion follows. ∎

Now we can prove the main auxiliary result we need for the proof of Theorem 4.1. It is a property of the space BV which could be interesting also in itself.

Theorem 4.10 *Let (u_h) be a sequence in $BV(\Omega;\mathbf{R}^N)$ and let $u \in BV(\Omega;\mathbf{R}^N)$ with $\|u_h - u\|_1 \to 0$ and $\mathcal{E}(u_h) \to \mathcal{E}(u)$.*

Then (u_h) is strongly convergent to u in $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$.

Proof. By Lemma 4.2 we may find $v_h \in C_c^\infty(\Omega; \mathbf{R}^N)$ with

$$\|v_h - u_h\|_1 < \frac{1}{h}\,, \quad \|v_h - u_h\|_{\frac{n}{n-1}} < \frac{1}{h}\,, \quad |\mathcal{E}(v_h) - \mathcal{E}(u_h)| < \frac{1}{h}\,.$$

Therefore it is sufficent to treat the case in which $u_h \in C_c^\infty(\Omega; \mathbf{R}^N)$.

By contradiction, up to a subsequence we may assume that there exists $\varepsilon > 0$ such that $\|u_h - u\|_{\frac{n}{n-1}} \geq \varepsilon$. Let $\tilde{c}$ be a constant such that $\|w\|_{\frac{n}{n-1}} \leq \tilde{c}\|w\|_{BV}$ for any $w \in BV(\Omega; \mathbf{R}^N)$ (see [24, Theorem 1.28]). According to Lemma 4.9, let $k \in \mathbf{N}$ be such that

$$\|R_k \circ u\|_{\frac{n}{n-1}} < \frac{\varepsilon}{2}\,, \quad \liminf_{h\to\infty} \|R_k \circ u_h\|_{\frac{n}{n-1}} \leq \tilde{c}\, \liminf_{h\to\infty} \|R_k \circ u_h\|_{BV} < \frac{\varepsilon}{2}\,.$$

Then we have

$$\|u_h - u\|_{\frac{n}{n-1}} \leq \|R_k \circ u_h\|_{\frac{n}{n-1}} + \|T_k \circ u_h - T_k \circ u\|_{\frac{n}{n-1}} + \|R_k \circ u\|_{\frac{n}{n-1}}\,. \tag{4.11}$$

Since $T_k \circ u_h \to T_k \circ u$ in $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$ as $h \to \infty$, passing to the lower limit in (4.11) we get

$$\liminf_{h\to\infty} \|u_h - u\|_{\frac{n}{n-1}} < \varepsilon\,,$$

whence a contradiction. ∎

Proof of Theorem 4.1. It is well known that $\mathcal{E}$ satisfies condition $(\mathcal{E}_1)$. Conditions $(\mathcal{E}_2)$ are an immediate consequence of Lemma 4.4. From (e) of Lemma 4.3 and Rellich's Theorem (see [24, Theorem 1.19]) it follows that $\mathcal{E}$ satisfies condition $(\mathcal{E}_3)$. To prove $(\mathcal{E}_4)$, let (u_h) be a sequence in $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$ weakly convergent to $u \in BV(\Omega; \mathbf{R}^N)$ such that $\mathcal{E}(u_h)$ converges to $\mathcal{E}(u)$. Again by (e) of Lemma 4.3 and Rellich's Theorem we deduce that (u_h) is strongly convergent to u in $L^1(\Omega; \mathbf{R}^N)$. Then the assertion follows from Theorem 4.10. ∎

5 A result of Clark type

Let $n \geq 2$ and Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary, let $\Psi : \mathbf{R}^{nN} \to \mathbf{R}$ be an even convex function satisfying (Ψ) and let $G : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a function satisfying (G_1), (G_2), (G_3'), (G_4') with $p = \frac{n}{n-1}$ and the following conditions:

$$\begin{cases} \text{there exist } \tilde{a} \in L^1(\Omega) \text{ and } \tilde{b} \in L^n(\Omega) \text{ such that} \\ G(x,s) \geq -\tilde{a}(x) - \tilde{b}(x)|s| \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,; \end{cases} \tag{5.1}$$

$$\lim_{|s|\to\infty} \frac{G(x,s)}{|s|} = +\infty \qquad \text{for a.e. } x \in \Omega\,; \tag{5.2}$$

$$\{s \longmapsto G(x,s)\} \text{ is even for a.e. } x \in \Omega\,. \tag{5.3}$$

Finally, define $\mathcal{E}$ as in Section 4. The main result of this section is:

Theorem 5.4 *For every $k \in \mathbf{N}$ there exists Λ_k such that for any $\lambda \geq \Lambda_k$ the problem*

$$\begin{cases} u \in BV(\Omega; \mathbf{R}^N) \\ \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x, u; v-u)\, dx \geq \lambda \int_\Omega \frac{u}{\sqrt{1+|u|^2}} \cdot (v-u)\, dx \quad \forall v \in BV(\Omega; \mathbf{R}^N) \end{cases}$$

admits at least k pairs $(u, -u)$ of distinct solutions.

For the proof we need the following

Lemma 5.5 *Let (u_h) be a bounded sequence in $L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N)$, which is convergent a.e. to u, and let (ϱ_h) be a positively divergent sequence of real numbers.*

Then we have

$$\lim_h \int_\Omega \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx = +\infty \quad \text{if } u \neq 0\,,$$

$$\liminf_h \int_\Omega \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx \geq 0 \quad \text{if } u = 0\,.$$

Proof. If $u = 0$, the assertion follows directly from (5.1). If $u \neq 0$, we have

$$\int_\Omega \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx \geq \int_{\{u \neq 0\}} \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx - \frac{1}{\varrho_h} \int_{\{u=0\}} \tilde{a}\, dx - \int_{\{u=0\}} \tilde{b} |u_h|\, dx\,.$$

From (5.1), (5.2) and Fatou's Lemma, we deduce that

$$\lim_h \int_{\{u \neq 0\}} \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx = +\infty\,,$$

whence the assertion. ■

Proof of Theorem 5.4. First of all, set

$$\widetilde{G}(x, s) = G(x, s) - \lambda(\sqrt{1+|s|^2} - 1)\,.$$

It is easy to see that also $\widetilde{G}$ satisfies (G_1), (G_2), (G_3'), (G_4'), (5.1), (5.2), (5.3) and that

$$\widetilde{G}^\circ(x, s; \hat{s}) = G^\circ(x, s; \hat{s}) - \lambda \frac{s}{\sqrt{1+|s|^2}} \cdot \hat{s}\,.$$

Now define a lower semicontinuous functional $f : L^{\frac{n}{n-1}}(\Omega; \mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ by

$$f(u) = \mathcal{E}(u) + \int_\Omega \widetilde{G}(x, u)\, dx\,.$$

Then f is even by (5.3) and satisfies condition $(epi)_c$ by Theorem 3.11. We claim that

(5.6) $$\lim_{\|u\|_{\frac{n}{n-1}} \to \infty} f(u) = +\infty\,.$$

To prove it, let (u_h) be a sequence in $BV(\Omega;\mathbf{R}^N)$ with $\|u_h\|_{\frac{n}{n-1}} = 1$ and let $\varrho_h \to +\infty$. By (e) of Lemma 4.3 there exist $\tilde{c} > 0$ and $\tilde{d} > 0$ such that

$$\forall u \in BV(\Omega;\mathbf{R}^N): \quad \mathcal{E}(u) \geq \tilde{d}\Big(\|u\|_{BV} - \tilde{c}\mathcal{L}^n(\Omega)\Big).$$

If $\|u_h\|_{BV} \to +\infty$, it readily follows from (5.1) that $f(\varrho_h u_h) \to +\infty$. Otherwise, up to a subsequence, u_h is convergent a.e. and the assertion follows from the previous Lemma and the inequality

$$f(\varrho_h u_h) \geq \varrho_h \left[\tilde{d}\left(\|u_h\|_{BV} - \frac{\tilde{c}}{\varrho_h}\mathcal{L}^n(\Omega)\right) + \int_\Omega \frac{\widetilde{G}(x,\varrho_h u_h)}{\varrho_h}\,dx\right].$$

Since f is bounded below on bounded subsets of $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$, it follows from (5.6) that f is bounded below on all $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$; furthermore, it also turns out from (5.6) that any $(PS)_c$ sequence is bounded, hence f satisfies $(PS)_c$ by Theorem 3.12.

Finally, let $k \geq 1$, let $w_1, \ldots, w_k$ be linearly independent elements of $BV(\Omega;\mathbf{R}^N)$ and let $\psi : S^{k-1} \to L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ be the odd continuous map defined by

$$\psi(\xi) = \sum_{j=1}^{k} \xi_j w_j.$$

Because of (G_3'), it is easily seen that

$$\sup\left\{\mathcal{E}(u) + \int_\Omega G(x,u)\,dx : u \in \psi(S^{k-1})\right\} < +\infty$$

and

$$\inf\left\{\int_\Omega (\sqrt{1+|u|^2} - 1)\,dx : u \in \psi(S^{k-1})\right\} > 0.$$

Therefore there exists $\Lambda_k > 0$ such that $\sup\limits_{\xi \in S^{k-1}} f(\psi(\xi)) < 0$ whenever $\lambda \geq \Lambda_k$.

Applying Theorem 2.12, it follows that f admits at least k pairs $(u_k, -u_k)$ of critical points. Therefore, by Theorem 2.16, for any u_k it is possible to apply Theorem 3.6 (with $\widetilde{G}$ instead of G), whence the assertion. ∎

6 A superlinear potential

Let $n \geq 2$ and Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary, let $\Psi : \mathbf{R}^{nN} \to \mathbf{R}$ be an even convex function satisfying (Ψ) and let $G : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a function satisfying (G_1), (G_2), (G_3'), (G_4'), (5.3) with $p = \frac{n}{n-1}$ and the following condition:

$$(6.1)\begin{cases} \text{there exist } q > 1 \text{ and } R > 0 \text{ such that} \\ G^\circ(x,s;s) \leq qG(x,s) < 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N \text{ with } |s| \geq R. \end{cases}$$

Define $\mathcal{E}$ as in section 4 and an even lower semicontinuous functional $f : L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N) \to \mathbf{R} \cup \{+\infty\}$ by

$$f(u) = \mathcal{E}(u) + \int_\Omega G(x,u)\,dx.$$

Theorem 6.2 *There exists a sequence* (u_h) *of solutions of the problem*

$$\begin{cases} u \in BV(\Omega;\mathbf{R}^N) \\ \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u;v-u)\,dx \geq 0 \quad \forall v \in BV(\Omega;\mathbf{R}^N) \end{cases}$$

with $f(u_h) \to +\infty$.

Proof. According to (3.3), we have

$$|s| < R \quad \Longrightarrow \quad |G^\circ(x,s;s)| \leq \alpha_R(x)|s|\,.$$

Combining this fact with (6.1) and (G_3'), we deduce that there exists $a_0 \in L^1(\Omega)$ such that

$$G^\circ(x,s;s) \leq qG(x,s) + a_0(x) \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,. \tag{6.3}$$

Moreover, from (6.1) and Lebourg's Theorem [14] it follows that for every $s \in \mathbf{R}^N$ with $|s| = 1$ the function $\{t \to t^{-q}G(x,ts)\}$ is nonincreasing on $[R,+\infty[$. Taking into account (G_3') and possibly substituting a_0 with another function in $L^1(\Omega)$, we deduce that

$$G(x,s) \leq a_0(x) - b_0(x)|s|^q \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,, \tag{6.4}$$

where

$$b_0(x) = \inf_{|s|=1}(-R^{-q}G(x,Rs)) > 0 \qquad \text{for a.e. } x \in \Omega\,.$$

Finally, since $\{\hat{s} \to G^\circ(x,s;\hat{s})\}$ is a convex function vanishing at the origin, we have $G^\circ(x,s;s) \geq -G^\circ(x,s;-s)$. Combining (6.3) with (G_4'), we deduce that for every $\varepsilon > 0$ there exists $\tilde{a}_\varepsilon \in L^1(\Omega)$ such that

$$G(x,s) \geq -\tilde{a}_\varepsilon(x) - \varepsilon|s|^{\frac{n}{n-1}} \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbf{R}^N\,. \tag{6.5}$$

By Theorem 3.11 we have that f satisfies $(epi)_c$ for any $c \in \mathbf{R}$ and that $|d_{\mathbf{Z}_2}\mathcal{G}_f|\,(0,\lambda) = 1$ for any $\lambda > f(0)$.

We also recall that, since Ψ is Lipschitz continuous, there exists $M \in \mathbf{R}$ such that

$$(q+1)\Psi(\xi) - \Psi(2\xi) \geq \frac{q-1}{2}\Psi(\xi) - M\,, \tag{6.6}$$

$$(q+1)\Psi^\infty(\xi) - \Psi^\infty(2\xi) \geq \frac{q-1}{2}\Psi^\infty(\xi) \tag{6.7}$$

(see also [36]).

We claim that f satifies the condition $(PS)_c$ for every $c \in \mathbf{R}$. Let (u_h) be a $(PS)_c$-sequence for f. By Theorem 2.16 there exists a sequence (u_h^*) in $L^n(\Omega;\mathbf{R}^N)$ with $u_h^* \in \partial f(u_h)$ and $\|u_h^*\|_n \to 0$. According to Theorem 3.6 and (6.3), we have

$$\begin{aligned} \mathcal{E}(2u_h) \;&\geq\; \mathcal{E}(u_h) - \int_\Omega G^\circ(x,u_h;u_h)\,dx + \int_\Omega u_h^* \cdot u_h\,dx \geq \\ &\geq\; \mathcal{E}(u_h) - q\int_\Omega G(x,u_h)\,dx + \int_\Omega u_h^* \cdot u_h\,dx - \int_\Omega a_0(x)\,dx\,. \end{aligned}$$

By the definition of f, it follows

$$qf(u_h) + \|u_h^*\|_n \|u_h\|_{\frac{n}{n-1}} + \int_\Omega a_0(x)\,dx \ge (q+1)\mathcal{E}(u_h) - \mathcal{E}(2u_h)\,.$$

Finally, applying (6.6) and (6.7) we get

$$qf(u_h) + \|u_h^*\|_n \|u_h\|_{\frac{n}{n-1}} + \int_\Omega a_0(x)\,dx \ge \frac{q-1}{2}\mathcal{E}(u_h) - M\mathcal{L}^n(\Omega)\,.$$

By (e) of Lemma 4.3 we deduce that (u_h) is bounded in $BV(\Omega;\mathbf{R}^N)$, hence in $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$. Applying Theorem 3.12 we get that (u_h) admits a strongly convergent subsequence and $(PS)_c$ follows.

By [36, Lemma 3.8], there exist a strictly increasing sequence (W_h) of finite-dimensional subspaces of $BV(\Omega;\mathbf{R}^N) \cap L^\infty(\Omega;\mathbf{R}^N)$ and a strictly decreasing sequence (Z_h) of closed subspaces of $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ such that $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N) = W_h \oplus Z_h$ and $\bigcap\limits_{h=0}^{\infty} Z_h = \{0\}$. By (e) of Lemma 4.3 there exists $\varrho > 0$ such that

$$\forall u \in L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N): \qquad \|u\|_{\frac{n}{n-1}} = \varrho \quad \Longrightarrow \quad \mathcal{E}(u) \ge 1\,.$$

We claim that

$$\lim_h \Big(\inf\{f(u) : u \in Z_h, \|u\|_{\frac{n}{n-1}} = \varrho\}\Big) > f(0)\,.$$

Actually, assume by contradiction that (u_h) is a sequence with $u_h \in Z_h$, $\|u_h\|_{\frac{n}{n-1}} = \varrho$ and

$$\limsup_h f(u_h) \le f(0)\,.$$

Taking into account (G_3') and Lemma 4.3, we deduce that $(\mathcal{E}(u_h))$ is bounded, so that (u_h) is bounded in $BV(\Omega;\mathbf{R}^N)$. Therefore, up to a subsequence, (u_h) is convergent a.e. to 0. From (6.5) it follows that

$$\liminf_h \int_\Omega \Big(G(x,u_h) + \varepsilon|u_h|^{\frac{n}{n-1}}\Big)\,dx \ge \int_\Omega G(x,0)\,dx\,,$$

hence

$$\liminf_h \int_\Omega G(x,u_h)\,dx \ge \int_\Omega G(x,0)\,dx$$

by the boundedness of (u_h) in $L^{\frac{n}{n-1}}(\Omega;\mathbf{R}^N)$ and the arbitrariness of ε. Therefore

$$\limsup_h \mathcal{E}(u_h) \le \mathcal{E}(0) = 0$$

which contradicts the choice of ϱ.

Now, fix $\overline{h}$ with

$$\inf\{f(u) : u \in Z_{\overline{h}}, \|u\|_{\frac{n}{n-1}} = \varrho\}\Big) > f(0)$$

and set $Z = Z_{\overline{h}}$ and $V_h = W_{\overline{h}+h}$. Then Z satisfies assumption (a) of Theorem 2.13 for some $\alpha > f(0)$.

Finally, since V_h is finite-dimensional,

$$\|u\|_G := \left(\int_\Omega b_0 |u|^q dx\right)^{\frac{1}{q}}$$

is a norm on V_h equivalent to the norm of $BV(\Omega; \mathbf{R}^N)$. Then, combining (6.4) with (d) of Lemma 4.3, we see that also assumption (b) of Theorem 2.13 is satisfied.

Therefore there exists a sequence (u_h) of critical points for f with $f(u_h) \to +\infty$ and, by Theorems 2.16 and 3.6, the result follows. ■

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Nonlinear eigenvalue problems arising in earthquake initiation

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Abstract

We study a symmetric nonlinear eigenvalue problem arising in earthquake initiation and we establish the existence of infinitely many solutions. Under the effect of an arbitrary perturbation, we prove that the number of solutions becomes greater and greater if the perturbation tends to zero with respect to a prescribed topology. Our approach is based on non-smooth critical point theories in the sense of De Giorgi and Degiovanni.

Keywords: nonlinear eigenvalue problem, Lusternik-Schnirelmann theory, symmetry, perturbation, earthquake initiation.

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1 Introduction

The minimax method has been used intensively in constructing critical points for functionals defined on Hilbert or Banach spaces as solutions of nonlinear partial differential equations or boundary value problems for inequality problems. In particular, when the problems possess symmetry, one can construct multiple critical points by the minimax method. This is the general Lusternik-Schnirelmann type theory (see [2, 18, 19, 21, 23, 25]). When an order structure is present, one can also use fixed point theory, topological degree arguments or variational methods to construct solutions of differential equations or variational inequalities (see [1, 6, 7, 12, 14]). However, little work has been done for invariant energy functionals under group actions when one expects to obtain multiplicity of critical points.

The main purpose of this paper is to consider a concrete nonlinear eigenvalue variational inequality arising in earthquake initiation and to establish, in the setting of the non-smooth Lusternik-Schnirelmann theory, the existence of infinitely many solutions. The main novelty in our framework is the presence of the convex cone of functions with non-negative jump across an internal boundary which is composed of a finite number of bounded connected arcs.

Under some natural assumptions, we prove the existence of infinitely many solutions, as well as further properties of eigensolutions and eigenvalues. Since the associated energy functional is included neither in the theory of monotone operators, nor in their Lipschitz perturbations, we employ the notion of lower subdifferential which is originally due to De Giorgi.

Next, we are concerned with the study of the effect of a small non-symmetric perturbation and we prove that the number of solutions of the perturbed problem becomes greater and greater if the perturbation tends to zero with respect to an appropriate topology. Our proof relies on powerful methods from algebraic topology developed in Krasnoselski [18] combined with adequate tools in the sense of the Degiovanni non-smooth critical point theory (see [8, 12, 13]).

2 Physical motivation

Consider, as in [3, 5, 10, 16, 27], the anti-plane shearing on a system of finite faults under a slip-dependent friction in an homogeneous linear elastic domain . Let $\Omega \subset \mathbb{R}^2$ be a domain, not necessarily bounded, containing a finite number of cuts. Its boundary $\partial\Omega$ is supposed to be smooth and divided into two disjoint parts: the exterior boundary $\Gamma_d = \partial\bar{\Omega}$ and the internal one Γ composed by N_f bounded connected arcs $\Gamma_f^i, i = 1, .., N_f$, called cracks or faults. We suppose that the displacement field is 0 in directions Ox and Oy and that u_z does not depend on z. The displacement is therefore denoted simply by $w = w(t, x, y)$. The elastic medium has the shear rigidity G, the density ρ and the shear velocity $c = \sqrt{G/\rho}$. The non-vanishing shear stress components are $\sigma_{zx} = \tau_x^\infty + G\partial_x w$, $\sigma_{zy} = \tau_y^\infty + G\partial_y w$, and $\sigma_{xx} = \sigma_{yy} = -S$ ($S > 0$ is the normal stress on the fault plane). We look for $w : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ solution of the wave equation :

$$\partial_{tt} w(t) = c^2 \Delta w(t) \quad \text{in} \quad \Omega, \tag{1}$$

with the boundary condition :

$$w(t) = 0 \ \text{ on } \quad \Gamma_d, \tag{2}$$

On Γ we denote by $[\ \ ]$ the jump across Γ, (i.e. $[w] = w^+ - w^-$) and by $\partial_n = \nabla \cdot n$ the corresponding normal derivative with the unit normal n outwards the positive side. On the contact zone Γ we have $[\partial_n w] = 0$ and a slip dependent friction law (introduced in the geophysical context of earthquakes modelling) is assumed :

$$G\partial_n w(t) = -\mu(|[w(t)]|)S\text{sign}([\partial_t w(t)]) - q, \ \text{ if } \ [\partial_t w(t)] \neq 0, \tag{3}$$

$$|G\partial_n w(t) + q| \leq \mu(|[w(t)]|)S \ \text{ if } \ \partial_t [w(t)] = 0, \tag{4}$$

where $q = \tau_x^\infty n_x + \tau_y^\infty n_y$. The initial conditions are

$$w(0) = w_0, \quad \partial_t w(0) = w_1 \quad \text{in} \quad \Omega. \tag{5}$$

Any solution of the above problem let satisfies the following variational problem (VP): find $w : [0,T] \longrightarrow V$ such that

$$\int_\Omega \frac{1}{c^2}\partial_{tt}w(t)(v-\partial_t w(t)\ dx\ + \int_\Omega \nabla w(t)\cdot\nabla(v-\partial_t w(t))\ dx\ + \tag{6}$$
$$\int_\Gamma \frac{S}{G}\mu(|[w(t)]|)(|[v]|-|[\partial_t w(t)]|)\ d\sigma\ \geq \int_\Gamma \frac{1}{G}q([v]-[\partial_t w(t)])\ d\sigma,$$

for all $v \in V$, where

$$V = \{v \in H^1(\Omega)/v = 0 \ \text{ on } \ \Gamma_d\}. \tag{7}$$

The main difficulty in the study of the above evolution variational inequality is the non-monotone dependence of μ with respect to the slip $|[w]|$. However, in modelling unstable phenomena, as earthquakes, we have to expect "bad" mathematical properties of the operators involved in the abstract problem. The existence of a solution w of the following regularity

$$w \in W^{1,\infty}(0,T,V)\cap W^{2,\infty}(0,T,L^2(\Omega)). \tag{8}$$

in the two-dimensional case was recently proved by Ionescu et al. [17]. The uniqueness was obtained only in the one-dimensional case.

Since our intention is to study the evolution of the elastic system near an unstable equilibrium position, we shall suppose that $q = \mu(0)S$. We remark that $w \equiv 0$ is an equilibrium solution of (6), and w_0, w_1 may be considered as small perturbations of it.

For simplicity, let us assume in the following that the friction law is homogeneous on the fault plane having the form of a piecewise linear function (see [24]) :

$$\mu(x,u) = \mu_s - \frac{\mu_s-\mu_d}{2D_c}u \ \text{ if } \ u \leq 2D_c, \ \ \mu(x,u) = \mu_d \ \text{ if } \ u > 2D_c, \tag{9}$$

where u is the relative slip, μ_s and μ_d ($\mu_s > \mu_d$) are the static and dynamic friction coefficients, and D_c is the critical slip. This piecewise linear function is a reasonable approximation of the experimental observations reported by [22]. Since the initial perturbation (w_0, w_1) of the equilibrium ($w \equiv 0$) is small we have $[w(t,x))] \leq 2D_c$ for $t \in [0,T_c]$ for all $x \in \Gamma$, where T_c is a critical time for which the slip on the fault reaches the critical value $2D_c$ at least at one point. Hence for a first period $[0,T_c]$, called the *initiation phase*, we deal with a linear function μ.

Our aim is to analyze the evolution of the perturbation during this initial phase. That is why we are interested in the existence of solutions of the type

$$w(t,x) = \sinh(|\lambda|ct)u(x), \quad w(t,x) = \sin(|\lambda|ct)u(x) \tag{10}$$

during the initiation phase $t \in [0,T_c]$. If we put the above expression in (6) and we have in mind that from (9) we have $\mu(s) = \mu_s - (\mu_s-\mu_d)/(2D_c)s$ then we deduce that (u,λ^2) is the solution of the nonlinear eigenvalue problem

$$\left\{ \begin{array}{l} \text{find } u \in K \text{ and } \lambda^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u\cdot\nabla(v-u)dx - \beta\int_\Gamma [u]\ [v-u]d\sigma + \lambda^2\int_\Omega u(v-u)dx \geq 0, \end{array} \right. \tag{11}$$

for all $v \in K$, where K is the convex closed cone centered at the origin

$$K = \{v \in V;\ [v] \geq 0 \text{ on } \Gamma\}$$

and $\beta = (\mu_s - \mu_d)S/(2D_cG) > 0$. The first type of solution from (10) has an exponential growth in time and corresponds to $\lambda^2 > 0$. The second one has the same amplitude during the initiation phase and corresponds to $\lambda^2 < 0$.

The nonlinear eigenvalue problem (11) can be written as classical eigenvalue for the Laplace operator with Signorini-type boundary conditions :

$$\text{find } u : \Omega \to \mathbb{R} \text{ and } \lambda^2 \in \mathbb{R} \text{ such that}$$

$$\Delta u = \lambda^2 u \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_d, \tag{12}$$

$$[\partial_n u] = 0,\ [u] \geq 0,\ \partial_n u \geq 0,\ [u](\partial_n u - \beta[u]) = 0 \text{ on } \Gamma. \tag{13}$$

The linear case, that is equation (12) with the boundary condition

$$[\partial_n u] = 0,\ \partial_n u - \beta[u] = 0 \text{ on } \Gamma, \tag{14}$$

was analyzed in [9]. For bounded domains, they proved that the spectrum of (12),(14) consists of a decreasing and unbounded sequence of eigenvalues. The greatest one, λ_0^2, which may be positive, is showed to be an increasing function of the friction parameter β. Let us remark that if u is a solution of (12), (14) and $[u] \geq 0$ on Γ_f then u is a solution for (12), (13) too. For co-linear faults the first eigenfunction u_0, corresponding to λ_0^2 was found in numerical computations to be positive on Γ_f (see [9, 10]), hence the linear case was sufficient to give a good model for the initiation of instabilities. If the faults are not co-linear, then this condition is not anymore satisfied, that is the first eigenfunction of the linear problem has no physical significance. Hence, in modelling initiation of friction instabilities only the non-linear eigenvalue problem has to be considered. As it was reported in [28], where the case of two parallel faults was analyzed, there exits an important gap between the first eigenvalues of the linear and nonlinear problems.

3 The main results

Let Ω be a smooth, bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ as in the preceding section, that is, containing a finite number of cuts. The internal boundary is denoted by Γ and the exterior one by Γ_d. Denote by $\|\cdot\|$ the norm in the space V, as defined in (7), and by $\Lambda_0 : L^2(\Omega) \to L^2(\Omega)^*$ and $\Lambda_1 : V \to V^*$ the duality isomorphisms defined by

$$\Lambda_0 u(v) = \int_\Omega uv dx, \qquad \text{for any } u, v \in L^2(\Omega)$$

and

$$\Lambda_1 u(v) = \int_\Omega \nabla u \cdot \nabla v dx, \qquad \text{for any } u, v \in V.$$

In order to reformulate our problem, consider the Lipschitz map $\gamma = i \circ \eta : V \to L^2(\Gamma)$, where $\eta : V \to H^{1/2}(\Gamma)$ is the trace operator, $\eta(v) = [v]$ on Γ and

$i : H^{1/2}(\Gamma) \to L^2(\Gamma)$ is the embedding operator. Then the space $L^2(\Gamma)$ is compactly embedded in V through the operator γ (see [15]).

Thus problem (11) can be written, equivalently,

$$\left\{ \begin{array}{l} \text{find } u \in K \text{ and } \lambda^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla (v-u)dx + \int_\Gamma j'\left(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\right) d\sigma + \\ \displaystyle\lambda^2 \int_\Omega u(v-u)dx \geq 0, \quad \forall v \in K, \end{array} \right. \tag{15}$$

where

$$j : \mathbb{R} \to \mathbb{R} \qquad j(t) = -\frac{\beta}{2}t^2$$

and $j'(\cdot\, ;\, \cdot)$ stands for the Gâteaux directional derivative.

Due to the homogeneity of (15), we can reformulate this problem in terms of a constrained inequality problem as follows. For any fixed $r > 0$, set

$$M = \left\{ u \in V; \int_\Omega u^2 dx = r^2 \right\}.$$

Then M is a smooth manifold in the Hilbert space V. We shall study the problem

$$\left\{ \begin{array}{l} \text{find } u \in K \cap M \text{ and } \lambda^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla (v-u)dx + \int_\Gamma j'\left(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\right) d\sigma + \\ \displaystyle\lambda^2 \int_\Omega u(v-u)dx \geq 0, \quad \forall v \in K. \end{array} \right. \tag{16}$$

Our multiplicity result is

Theorem 3.1. *Problem (16) has infinitely many solutions (u, λ^2) and the set of eigenvalues $\{\lambda^2\}$ is bounded from above and its infimum equals to $-\infty$. Let $\lambda_0^2 = \sup\{\lambda^2\}$. Then there exists u_0 such that (u_0, λ_0^2) is a solution of (16). Moreover the function $\beta \longmapsto \lambda_0^2(\beta)$ is convex and the following inequality holds*

$$\int_\Omega |\nabla v|^2 \, dx + \lambda_0^2(\beta) \int_\Omega v^2 \, dx \geq \beta \int_\Gamma [v]^2 \, d\sigma, \qquad \forall v \in K. \tag{17}$$

Next, we study the effect of an arbitrary perturbation in problem (15). More precisely, we consider the problem

$$\left\{ \begin{array}{l} \text{find } u_\varepsilon \in K \text{ and } \lambda_\varepsilon^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla (v-u_\varepsilon)dx + \int_\Gamma (j' + \varepsilon g')\left(\gamma(u_\varepsilon(x)); \gamma(v(x)) - \gamma(u_\varepsilon(x))\right) d\sigma + \\ \displaystyle\lambda_\varepsilon^2 \int_\Omega u_\varepsilon(v-u_\varepsilon)dx \geq 0, \quad \forall v \in K, \end{array} \right. \tag{18}$$

where $\varepsilon > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with no symmetry hypothesis, but satisfying the growth assumption

$$\begin{array}{ll} \exists\, a > 0,\ \exists\, 2 \leq p \leq \dfrac{2(N-1)}{N-2} \text{ such that } |g(t)| \leq a(1+|t|^p) & , \text{ if } N \geq 3; \\ \exists\, a > 0,\ \exists\, 2 \leq p < +\infty \text{ such that } |g(t)| \leq a(1+|t|^p) & , \text{ if } N = 2. \end{array} \tag{19}$$

We prove that the number of solutions of problem (18) becomes greater and greater if the perturbation "tends" to zero. This is a very natural phenomenon that occurs often in concrete situations. We illustrate it with the following elementary example: consider on the real axis the equation $\sin x = 1/2$. This is a "symmetric" problem (due to the periodicity) with infinitely many solutions. Let us now consider an arbitrary non-symmetric "small" perturbation of the above equation, say $\sin x = 1/2 + \varepsilon x^2$. This equation has finitely many solutions, for any $\varepsilon \neq 0$. However, the number of solutions of the perturbed equation tends to infinity is the perturbation (that is, $|\varepsilon|$) becomes smaller and smaller.

More precisely, we have

Theorem 3.2. *For every positive integer n, there exists $\varepsilon_n > 0$ such that problem (18) has at least n distinct solutions $(u_\varepsilon, \lambda_\varepsilon^2)$ if $\varepsilon < \varepsilon_n$. There exists and is finite $\lambda_{0\varepsilon}^2 = \sup\{\lambda_\varepsilon^2\}$ and there exists $u_{0\varepsilon}$ such that $(u_{0\varepsilon}, \lambda_{0\varepsilon}^2)$ is a solution of (18). Moreover, $\lambda_{0\varepsilon}^2$ converges to λ_0^2 as ε tends to 0, where λ_0^2 was defined in Theorem 3.1.*

4 Auxiliary results

Several times in this paper we shall apply the following basic embedding inequality:

Proposition 4.1. *(Lemma 5.1 in [15]). Let $2 \le \alpha \le 2(N-1)/(N-2)$ if $N \ge 3$ and $2 \le \alpha < +\infty$ if $N = 2$. Then for $\beta = [(\alpha-2)N+2]/(2\alpha)$ if $N \ge 3$ or if $N = 2$ and $\alpha = 2$ and for all $(\alpha-1)/\alpha < \beta < 1$ if $N = 2$ and $\alpha > 2$, there exists $C = C(\beta)$ such that*

$$\left(\int_\Gamma |[u]|^\alpha d\sigma\right)^{1/\alpha} \le C\left(\int_\Omega u^2 dx\right)^{(1-\beta)/2}\left(\int_\Omega |\nabla u|^2 dx\right)^{\beta/2}, \qquad \textit{for any } \ u \in V. \quad (20)$$

An important role in our arguments in order to locate the solution of (16) will be played by the indicator function of M, that is,

$$I_M(u) = \begin{cases} 0 & , \quad \text{if } u \in M \\ +\infty & , \quad \text{if } u \in V \setminus M. \end{cases}$$

Then I_M is lower semicontinuous. However, since the natural energy functional associated to problem (16) is neither smooth nor convex, it is necessary to introduce a more general concept of gradient. We shall emply the following notion of lower subdifferential which is due to De Giorgi, Marino and Tosques [11]. The following definition agrees with the corresponding notions of gradient and critical point in the sense of Fréchet (for C^1 mappings), Clarke (for locally Lipschitz functionals) or in the sense of the convex analysis.

Definition 4.2. *Let X be a Banach space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary proper functional. Let $x \in D(f)$. The gradient of f at x is the (possibly empty) set*

$$\partial^- f(x) = \left\{\xi \in X^*;\ \liminf_{y \to x} \frac{f(y) - f(x) - \xi(y-x)}{\|y-x\|} \ge 0\right\}.$$

An element $\xi \in \partial^- f(x)$ is called a lower subdifferential of f at x.

Accordingly, we say that $x \in D(f)$ is a critical (lower stationary) point of f if $0 \in \partial^- f(x)$.

Then $\partial^- f(x)$ is a convex set. If $\partial^- f(x) \neq \emptyset$ we denote by $\mathrm{grad}^- f(x)$ the element of minimal norm of $\partial^- f(x)$, that is,

$$\mathrm{grad}^- f(x) = \min\{\|\xi\|_{X^*};\ \xi \in \partial^- f(x)\}.$$

This notion plays a central role in the statement of our basic compactness condition.

Definition 4.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary functional. We say that $(x_n) \subset D(f)$ is a Palais-Smale sequence if*

$$\sup_n |f(x_n)| < +\infty \qquad \text{and} \qquad \lim_{n\to\infty} \mathrm{grad}^- f(x_n) = 0.$$

The functional f is said to satisfy the Palais-Smale condition provided that any Palais-Smale sequence is relatively compact.

Remark 4.4. *(i) Definition 4.2 implies that if $g : X \to \mathbb{R}$ is Fréchet differentiable and $f : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary proper function then*

$$\partial^-(f+g)(x) = \left\{\xi + g'(x);\ \xi \in \partial^- f(x)\right\},$$

for any $x \in D(f)$.

(ii) Similary, if $f : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary proper functional and $g : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous then

$$\partial^-(f+g)(x) = \left\{\xi + g'(x);\ \xi \in \partial^- f(x)\right\},$$

for any $x \in D(f) \cap D(g)$.

As established in [7],

$$\partial^- I_M(u) = \{\lambda \Lambda_0 u;\ \lambda \in \mathbb{R}\} \subset L^2(\Omega)^* \subset V^*, \qquad \text{for any } u \in M. \tag{21}$$

In the proof of Theorems 3.1 and 3.2 we shall use several auxiliary notions and properties. For the convenience of the reader we recall them in what follows. For further details and proofs we refer to [12, 19, 21, 23, 26].

A topological space X is said to be *contractible* if the identity of X is homotopical to a constant map, that is, there exists $u_0 \in X$ and a continuous map $F : X \times [0,1] \to X$ such that

$$F(\cdot, 0) = \mathrm{Id}_X \qquad \text{and} \qquad F(\cdot, 1) = u_0.$$

A subset M of X is said to be *contractible* in X if there exists $u_0 \in X$ and a continuous map $F : M \times [0,1] \to X$ such that

$$F(\cdot, 0) = \mathrm{Id}_M \qquad \text{and} \qquad F(\cdot, 1) = u_0.$$

If A is a subset of X, we define the category of A in X as follows:

$\mathrm{Cat}_X(A) = 0$, if $A = \emptyset$.

$\mathrm{Cat}_X(A) = n$, if n is the smallest integer such that A can be covered by n closed sets which are contractible in X.

$\mathrm{Cat}_X(A) = +\infty$, otherwise.

Some basic properties of the notion of category are summarized in

Proposition 4.5. *The following properties hold true:*

(i) If $A \subset B \subset X$, then $\mathrm{Cat}_X(A) \leq \mathrm{Cat}_X(B)$.

(ii) $\mathrm{Cat}_X(A \cup B) \leq \mathrm{Cat}_X(A) + \mathrm{Cat}_X(B)$

(iii) Let $h : A \times [0,1] \to X$ be a continuous mapping such that $h(x,0) = x$ for every $x \in A$. If A is closed and $B = h(A,1)$, then $\mathrm{Cat}_X(A) \leq \mathrm{Cat}_X(B)$

Let (X,d) be a metric space. Consider $h : X \to \mathbb{R} \cup \{+\infty\}$ an arbitrary functional and set, as usually, $D(h) := \{u \in X;\ h(u) < +\infty\}$. We recall the following definitions which are due essentially to De Giorgi (see, e.g., De Giorgi, Marino and Tosques [11]).

Definition 4.6. *(i) For $u \in D(h)$ and $\rho > 0$, let $h_u(\rho) = \inf\{h(v);\ d(v,u) < \rho\}$. Then the number $-D_+ h_u(0)$ is called the slope of h at u, where D_+ denotes the right lower derivative.*

(ii) Let $I \subset \mathbb{R}$ be an arbitrary non-trivial interval and consider a curve $U : I \to X$. We say that U is a curve of maximal slope for h if the following properties hold true:

– U is continuous;

– $h \circ U(t) < +\infty$, for any $t \in I$;

– $d(U(t_2), U(t_1)) \leq \int_{t_1}^{t_2} \left[D_+ h_{U(t)}(0)\right]^2 dt$, for any t_1, $t_2 \in I$, $t_1 < t_2$;

– $h \circ U(t_2) - h \circ U(t_1) \leq -\int_{t_1}^{t_2} \left[D_+ h_{U(t)}(0)\right]^2 dt$, for any t_1, $t_2 \in I$, $t_1 < t_2$.

In what follows, X denotes a metric space, A is a subset of X and i stands for the inclusion map of A in X.

Definition 4.7. *(i) A map $r : X \to A$ is said to be a retraction if it is continuous, surjective and $r_{|A} = Id$.*

(ii) A retraction r is called a strong deformation retraction provided that there exists a homotopy $\zeta : X \times [0,1] \to X$ of $i \circ r$ and Id_X which satisfies the additional condition $\zeta(x,t) = \zeta(x,0)$, for any $(x,t) \in A \times [0,1]$.

(iii) The metric space X is said to be weakly locally contractible, if for every $u \in X$ there exists a neighbourhood U of u contractible in X.

For every $a \in \mathbb{R}$, denote

$$f^a = \{u \in X : f(u) \leq a\},$$

where $f : X \to \mathbb{R}$ is a continuous function.

Definition 4.8. *(i) Let $a, b \in \mathbb{R}$ with $a \leq b$. The pair (f^b, f^a) is said to be trivial provided that, for every neighbourhood $[a', a'']$ of a and $[b', b'']$ of b, there exists some closed sets A and B such that $f^{a'} \subseteq A \subseteq f^{a''}$, $f^{b'} \subseteq B \subseteq f^{b''}$ and such that A is a strong deformation retraction of B.*

(ii) A real number c is an essential value of f provided that, for every $\varepsilon > 0$ there exists $a, b \in (c - \varepsilon, c + \varepsilon)$ with $a < b$ such that the pair (f^b, f^a) is not trivial.

The following property of essential values is due to Degiovanni and Lancelotti (see [12], Theorem 2.6).

Proposition 4.9. *Let c be an essential value of f. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous function $g : X \to \mathbb{R}$ with*

$$\sup\{|g(u) - f(u)| : u \in X\} < \delta$$

admits an essential value in $(c - \varepsilon, c + \varepsilon)$.

For every $n \geq 1$, define

$$\Gamma_n = \{S \subset S_r;\ S \subset \mathcal{F}, \gamma(S) \geq n\},$$

where $\mathcal{F}$ is the class of closed symmetric subsets of the sphere S_r of radius r in a certain Banach space and $\gamma(S)$ represents the Krasnoselski genus of $S \in \Gamma_n$, that is, the smallest $k \in \mathbb{N} \cup \{+\infty\}$ for which there exists a continuous and odd map from S into $\mathbb{R}^k \setminus \{0\}$.

5 Proof of Theorem 3.1

Define

$$E = F + G : V \to \mathbb{R} \cup \{+\infty\},$$

where

$$F(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla u|^2 dx & , \quad \text{if } u \in K \\ +\infty & , \quad \text{if } u \notin K \end{cases}$$

and

$$G(u) = -\frac{\beta}{2} \int_\Gamma [\gamma(u(x))]^2 d\sigma.$$

Then $E + I_M$ is lower semicontinuous.

The following auxiliary result shows that $E + I_M$ is the canonical energy functional associated to problem (16).

Proposition 5.1. *If (u, λ^2) is a solution of problem (16) then $0 \in \partial^-(E + I_M)(u)$. Conversely, let u be a critical point of $E + I_M$ and denote $\lambda^2 = -2E(u)r^{-2}$. Then (u, λ^2) is a solution of problem (16).*

Proof. Let (u, λ^2) be a solution of problem (16). So, by the definition of the lower subdifferential,

$$-\lambda^2 u \in \partial^- E(u). \tag{22}$$

On the other hand,

$$\partial^-(E + I_M)(u) = \partial^- E(u) + \partial^- I_M(u), \qquad \text{for any } u \in K \cap M. \tag{23}$$

So, by (21) and (22), $0 \in \partial^-(E + I_M)(u)$.

Conversely, let $0 \in \partial^-(E + I_M)(u)$. Thus, by (21) and (23), there exists $\lambda^2 \in \mathbb{R}$ such that (u, λ^2) is a solution of problem (16). If we put $v = 0$ in (16) then we deduce $\lambda^2 r^2 \leq -2E(u)$ and for $v = 2u$ we get $\lambda^2 r^2 \geq -2E(u)$, that is $\lambda^2 = -2E(u)r^{-2}$. ■

The above result reduces our study to finding the critical points of $E + I_M$. In order to estimate the number of lower stationary points of this functional we shall apply a non-smooth version of the Lusternik-Schnirelmann theorem. For this purpose we need some preliminary results.

We first observe that a direct argument combined with Proposition 5.1 shows that problem (16) has at least one solution. Indeed, the associated energy functional is bounded from below. This follows directly by our basic inequality (20) since

$$(E + I_M)(u) \geq \frac{1}{2} \|u\|^2 - |\beta| \cdot \|[u]\|^2_{L^2(\Gamma)} \geq \frac{1}{2} \|u\|^2 - C \|u\| \geq C_0, \tag{24}$$

for any $u \in V$. So, by standard minimization arguments based on the compactness of the embedding $i \circ \eta : V \to L^2(\Gamma)$ we deduce that there exists a global minimum point $u_0 \in K \cap M$ of $E + I_M$. Let $\lambda_0^2 = -2E(u_0)/r^2$. Hence $0 \in \partial^-(E + I_M)(u_0)$ and (u_0, λ_0^2) is a solution of problem (16). Since for any eigenvalue λ^2 there exists $u \in K$ such that $\lambda^2 = -2E(u)r^{-2}$ we deduce that $\lambda_0^2 = \sup\{\lambda^2\}$.

The next step in our proof consists in showing that

Proposition 5.2. *The functional $E + I_M$ satisfies the Palais-Smale condition.*

Proof. Let (u_n) be an arbitrary Palais-Smale sequence of $E + I_M$. So, by (24), (u_n) is bounded in V. Thus, by the Rellich-Kondratchov theorem (see for instance [4]) and passing eventually at a subsequence,

$$u_n \rightharpoonup u \qquad \text{weakly in } V \tag{25}$$

$$u_n \to u \qquad \text{strongly in } L^2(\Omega) \tag{26}$$

$$u_n \to u \qquad \text{strongly in } L^2(\Gamma). \tag{27}$$

In particular, it follows that $u \in K \cap M$.

Using now the second information contained in the statement of the Palais-Smale condition and applying (21), we obtain a sequence (λ_n) of real numbers such that

$$\lim_{n\to\infty} \|E'(u_n) + \lambda_n \Lambda_0 u_n\|_{V^*} = 0. \tag{28}$$

On the other hand, by the compact embeddings $V \subset L^2(\Omega)$ and $V \subset L^2(\Gamma)$ and using (25)–(27), it follows that

$$E'(u_n) \to E'(u) \quad \text{and} \quad \Lambda_1 u_n \to \Lambda_1 u \qquad \text{in } V^*.$$

So, by (28), the sequence (λ_n) is bounded. Hence we can assume that, up to a subsequence, $\lambda_n \to \lambda$ as $n \to \infty$. Therefore $0 \in \partial^-(E + I_M)(u)$.

From (25) we get $\|u\| \le \liminf_{n\to\infty} \|u_n\|$, hence it follows that for concluding the proof it is enough to show that

$$\|u\| \ge \limsup_{n\to\infty} \|u_n\| . \tag{29}$$

But, since F is convex,

$$F(u) \ge F(u_n) + F'(u_n)(u - u_n).$$

It follows that

$$\begin{aligned} E(u) = \;& F(u) + G(u) \ge \limsup_{n\to\infty} \left(F(u_n) + F'(u_n)(u - u_n) + G(u_n)\right) = \\ & \limsup_{n\to\infty} \left(F(u_n) + F'(u_n)(u - u_n) + G(u_n) + G'(u_n)(u - u_n)\right) = \\ & \limsup_{n\to\infty} \left(F(u_n) + E'(u_n)(u - u_n)\right) + \lim_{n\to\infty} G(u_n). \end{aligned} \tag{30}$$

Using now $\lambda_n \to \lambda$ combined with (25)–(28), relation (30) yields

$$E(u) \ge \limsup_{n\to\infty} F(u_n) + G(u).$$

This inequality implies directly our claim (29), so the proof is completed. ■

Due to the symmetry of our problem (16), we can extend our study to the symmetric cone $(-K)$. More precisely, if (u, λ^2) is a solution of (16) then $u_0 := -u \in (-K) \cap M$ satisfies

$$\begin{aligned} &\int_\Omega \nabla u_0 \cdot \nabla(v - u_0)dx + \int_\Gamma j'\left(\gamma(u_0(x)); \gamma(v(x)) - \gamma(u_0(x))\right) d\sigma + \\ &\lambda^2 \int_\Omega u_0(v - u_0)dx \ge 0, \qquad \text{for all } v \in (-K). \end{aligned}$$

This means that we can extend the energy functional associated to problem (16) to the symmetric set $\widetilde{K} := K \cup (-K)$. We put, by definition,

$$\widetilde{E}(u) = \begin{cases} E(u) & , \quad \text{if } u \in K \\ E(-u) & , \quad \text{if } u \in (-K) \\ +\infty & , \quad \text{otherwise.} \end{cases}$$

We are interested from now in finding the lower stationary points of the extended energy functional $J := \widetilde{E} + I_M$.

We endow the set $\widetilde{K} \cap M$ with the graph metric of $\widetilde{E}$ defined by

$$d(u,v) = \|u-v\| + |\widetilde{E}(u) - \widetilde{E}(v)|, \qquad \text{for any } \; u,v \in \widetilde{K} \cap M.$$

Denote by $\mathcal{X}$ the metric space $(\widetilde{K} \cap M, d)$.

We are now in position to state the basic abstract result that we shall apply for concluding the proof of Theorem 3.1. More precisely, we use the following non-smooth variant of the Lusternik-Schnirelmann theory that we reformulate in terms of our energy functional J.

Theorem 5.3. *(Marino and Scolozzi [20]). Assume that J satisfies the following properties:*

(i) J is bounded from below;

(ii) J satisfies the Palais-Smale condition;

(iii) for any lower stationary point u of J there exists a neighbourhood of u in $\mathcal{X}$ which is contractible in $\mathcal{X}$;

(iv) there exists $\Theta : (\widetilde{K} \cap M) \times [0,\infty) \to \widetilde{K} \cap M$ such that $\Theta(\cdot, 0) = \mathrm{Id}$, $\Theta(u,\cdot)$ is a curve of maximal slope for J (with respect to the usual metric in V) and, moreover, the mapping $\Theta : \mathcal{X} \times [0,\infty) \to \mathcal{X}$ is continuous.

Then J has at least $\mathrm{Cat}_{\mathcal{X}}(\widetilde{K} \cap M)$ lower stationary points.

Moreover, if $\mathrm{Cat}_{\mathcal{X}}(\widetilde{K} \cap M) = +\infty$, then J does not have a maximum and

$$\sup\{J(u);\; u \in \widetilde{K} \cap M, 0 \in \partial^- J(u)\} = \sup\{J(u);\; u \in \widetilde{K} \cap M\}.$$

We have already proved (i) and (ii). Property (iii) is proved in a more general framework in De Giorgi, Marino and Tosques [11], while (iv) is deduced in Chobanov, Marino and Scolozzi [7]. So, using Theorem 5.3, it follows that for concluding the proof of Theorem 3.1 it remains to prove

Proposition 5.4. *We have*

$$\mathrm{Cat}_{\mathcal{X}}(\widetilde{K} \cap M) = +\infty. \tag{31}$$

Proof. Fix $\psi \in K \setminus \{0\}$ such that $\|\psi\|_{L^2(\Omega)} > r$ and let $(e_n)_{n \geq 1} \subset V$ be an orthonormal basis of $L^2(\Omega)$. Fix arbitrarily an integer $n \geq 1$ and denote

$$M^{(n)} = \{\sum_{i=1}^{n} \alpha_i e_i;\; \sum_{i=1}^{n} \alpha_i^2 = r^2\}.$$

As usually, we denote $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$, for any real number a. Define the mapping $\varphi_1 : M^{(n)} \times [0,1] \to V \setminus \{0\}$ by

$$\varphi_1(u,t) = (1-t)\left[(u-\psi)^+ - (u+\psi)^-\right] + P_K\left(\min\{\max(u,-\psi),\psi\}\right),$$

where P_K denotes the canonical projection onto K. Then

$$\varphi_1(u,1) \in K \qquad \text{and} \qquad \|\varphi_1(u,1)\|_{L^2} \leq \|u\|_{L^2} \leq r.$$

We also define $\varphi_2 : (\widetilde{K} \setminus \{0\}) \times [0,1) \to \widetilde{K} \setminus \{0\}$ by

$$\varphi_2(u,t) = \min \left[\max \left(\frac{1}{1-t}\, u, -\psi \right), \psi \right].$$

Fix arbitrarily $u \in \varphi_1(M^{(n)}, 1)$. Then

$$\lim_{t \nearrow 1} \|\varphi_2(u,t)\|_{L^2} = \|\psi\|_{L^2} > r.$$

The compactness of $\varphi_1(M^{(n)}, 1)$ implies that there exists $t_0 \in (0,1)$ such that

$$\|\varphi_2(u,t)\|_{L^2} > r \qquad \forall t \in [t_0, 1),\ \forall u \in \varphi_1(M^{(n)}, 1).$$

Let P be the canonical projection of V onto the closed ball of radius r in $L^2(\Omega)$ centered at the origin. Define the map $\Phi : M^{(n)} \times [0, 1+t_0] \to V \setminus \{0\}$ by

$$\Phi(u,t) = \begin{cases} \varphi_1(u,t) & , \quad \text{if } (u,t) \in M^{(n)} \times [0,1] \\ P\left(\varphi_2(\varphi_1(u,1), t-1)\right) & , \quad \text{if } (u,t) \in M^{(n)} \times [0, 1+t_0] \end{cases}$$

Then $\Phi(u,0) = 0$ and $\Phi(u, 1+t_0) \in M$. Since $\Phi(\cdot, t)$ is odd and continuous from $L^2(\Omega)$ in the L^2-topology, it follows by Proposition 4.5 that

$$n \le \mathrm{Cat}_{L^2}(M^{(n)}) \le \mathrm{Cat}_{L^2}\left(\Phi(M^{(n)}, 1+t_0)\right) \le \mathrm{Cat}_{H_0^1}\left(\Phi(M^{(n)}, 1+t_0)\right).$$

Since the set $\Phi(M^{(n)}, 1+t_0)$ is compact in V and the topology of $\mathcal{X}$ is stronger than the H_0^1-topology, we obtain

$$n \le \mathrm{Cat}_{H_0^1}\left(\Phi(M^{(n)}, 1+t_0)\right) \le \mathrm{Cat}_{\mathcal{X}}\left(\Phi(M^{(n)}, 1+t_0)\right) \le \mathrm{Cat}_{\mathcal{X}}(\widetilde{K} \cap M).$$

This completes the proof of Proposition 5.4. ∎

Proof of Theorem 3.1 completed. Until now, using Theorem 5.3, we have established that problem (16) admits infinitely many solutions (u, λ^2). We first observe that the set of eigenvalues is bounded from above. Indeed, if (u, λ^2) is a solution of our problem then choosing $v = 0$ in (16) and using (20), it follows that

$$\lambda^2 r^2 \le -2\|u\|^2 + \frac{\beta}{2}\, \|u\|^2_{L^2(\Gamma)} \le C,$$

where C does not depend on u.

It remains to prove that

$$\inf\{\lambda^2;\ \lambda^2 \text{ is an eigenvalue of (16)}\} = -\infty.$$

For this purpose, it is sufficient to show that

$$\sup\{J(u);\ u \in \widetilde{K} \cap M\} = +\infty.$$

But this follows directly from (20) and

$$\sup_{u\in\widetilde{K}\cap M}\int_\Omega |\nabla u|^2 dx = +\infty.$$

In order to prove the last part of the theorem we remark that $-\lambda_0$, as a function of β, is the upper bound of a family of affine functions

$$-\lambda_0^2(\beta) = \inf_{v\in K\cap M}\frac{1}{r^2}\left\{\int_\Omega |\nabla v|^2\, dx - \beta\int_\Gamma [v]^2\, d\sigma\right\}, \tag{32}$$

hence it is a concave function. Thus $\beta \longmapsto \lambda_0^2(\beta)$ is convex and (17) yields.

This concludes the proof of Theorem 3.1. ■

6 Proof of Theorem 3.2

We shall establish the multiplicity result with respect to a prescribed level of energy. More precisely, let us fix $r > 0$. Consider the manifold

$$N = \left\{u \in V;\ \int_\Gamma [u]^p d\sigma = r^p\right\},$$

where p is as in (19).

We reformulate problem (18) as follows:

$$\left\{\begin{array}{l} \text{find } u_\varepsilon \in K\cap N \text{ and } \lambda_\varepsilon^2 \in \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla(v-u_\varepsilon)dx + \int_\Gamma (j' + \varepsilon g')\left(\gamma(u_\varepsilon(x)); \gamma(v(x)) - \gamma(u_\varepsilon(x))\right) d\sigma + \\ \displaystyle\lambda_\varepsilon^2 \int_\Omega u_\varepsilon(v-u_\varepsilon)dx \geq 0, \quad \forall v \in K. \end{array}\right. \tag{33}$$

We start with the preliminary result

Lemma 6.1. *There exists a sequence (b_n) of essential values of E such that $b_n \to +\infty$ as $n\to\infty$.*

Proof. For any $n \geq 1$, set $a_n = \inf_{S\in\Gamma_n}\sup_{u\in S} E(u)$, where Γ_n is the family of compact subsets of $K\cap N$ of the form $\varphi(S^{n-1})$, with $\varphi : S^{n-1} \to K\cap N$ continuous and odd. The function E restricted to $K\cap N$ is continuous, even and bounded from below. So, by Theorem 2.12 in [12], it is sufficient to prove that $a_n \to +\infty$ as $n\to\infty$. But, by Proposition 5.2, the functional E restricted to $K\cap N$ satisfies the Palais-Smale condition. So, taking into account Theorem 3.5 in [8] and Theorem 3.9 in [12], we deduce that the set E^c has finite genus for any $c\in\mathbb{R}$. Using now the definition of the genus combined with the fact that $K\cap N$ is a weakly locally contractible metric space, we deduce that $a_n \to +\infty$. This completes our proof. ■

The canonical energy associated to problem (33) is the functional J restricted to $K \cap N$, where $J = E + \Phi$ and Φ is defined by

$$\Phi(u) = \varepsilon \int_\Gamma g(\gamma(u(x)))d\sigma.$$

A straightforward computation with the same arguments as in the proof of Proposition 5.1 shows that if u is a lower stationary point of J then there exists $\lambda^2 \in \mathbb{R}$ such that (u, λ^2) is a solution of problem (33). In virtue of this result, it is sufficient for concluding the proof of Theorem 3.2 to show that the functional J has at least n distinct critical values, provided that $\varepsilon > 0$ is sufficiently small. We first prove that J is a small perturbation of E. More precisely, we have

Lemma 6.2. *For every $\eta > 0$, there exists $\delta = \delta_\eta > 0$ such that $\sup_{u \in K \cap N} |J(u) - E(u)| \leq \eta$, provided that $\varepsilon \leq \delta$.*

Proof. We have

$$|J(u) - E(u)| = |\Phi(u)| \leq \varepsilon \int_\Gamma |g(\gamma(u(x)))| \, d\sigma.$$

So, by (19) and Proposition 4.1,

$$|J(u) - E(u)| \leq \varepsilon\, a \int_\Gamma (1 + [u(x)]^p) \, d\sigma \leq C\varepsilon \leq \eta,$$

if ε is sufficiently small. ■

By Lemma 6.1, there exists a sequence (b_n) of essential values of $E_{|K \cap N}$ such that $b_n \to +\infty$. Without loss of generality we can assume that $b_i < b_j$ if $i < j$. Fix an integer $n \geq 1$ and choose $\varepsilon_0 > 0$ such that $\varepsilon_0 < 1/2 \min_{2 \leq i \leq n}(b_i - b_{i-1})$. Applying now Proposition 4.9, we obtain that for any $1 \leq j \leq n$, there exists $\eta_j > 0$ such that if $\sup_{K \cap N} |J(u) - E(u)| < \eta_j$ then $J_{|K \cap N}$ has an essential value $c_j \in (b_j - \varepsilon_0, b_j + \varepsilon_0)$. So, by Lemma 6.2 applied for $\eta = \min\{\eta_1, \ldots, \eta_n\}$, there exists $\delta_n > 0$ such that $\sup_{K \cap N} |J(u) - E(u)| < \eta$, provided that $\varepsilon \leq \delta_n$. This shows that the energy functional J has at least n distinct essential values $c_1, \ldots, c_n$ in $(b_1 - \varepsilon_0, b_n + \varepsilon_0)$.

The next step consists in showing that $c_1, \ldots, c_n$ are critical values of $J_{|K \cap N}$. Arguing by contradiction, let us suppose that c_j is not a critical value of $J_{|K \cap N}$. We show in what follows that

(A_1) There exists $\bar{\delta} > 0$ such that $J_{|K \cap N}$ has no critical value in $(c_j - \bar{\delta}, c_j + \bar{\delta})$.

(A_2) For every $a, b \in (c_j - \bar{\delta}, c_j + \bar{\delta})$ with $a < b$, the pair $(J^b_{|K \cap N}, J^a_{|K \cap N})$ is trivial.

Suppose, by contradiction, that (A_1) is no valid. Then there exists a sequence (d_k) of critical values of $J_{|K \cap N}$ with $d_k \to c_j$ as $k \to \infty$. Since d_k is a critical value, it follows that there exists $u_k \in K \cap N$ such that

$$J(u_k) = d_k \quad \text{and} \quad 0 \in \partial^- J(u_k).$$

Using now the fact that J satisfies the Palais-Smale condition at the level c_j, it follows that, up to a subsequence, (u_k) converges to some $u \in K \cap N$ as $k \to \infty$. So, by the continuity of J and the lower semicontinuity of grad $J(\,\cdot\,)$, we obtain $J(u) = c_j$ and $0 \in \partial^- J(u)$, which contradicts the initial assumption on c_j.

Let us now prove assertion (A_2). For this purpose we apply the Noncritical Point Theorem (see [8], Theorem 2.15]). So, there exists a continuous map $\chi : (K \cap N) \times [0,1] \to K \cap N$ such that

$$\begin{array}{l} \chi(u,0) = u, \quad J(\chi(u,t)) \leq J(u), \\ J(u) \leq b \Rightarrow J(\chi(u,1)) \leq a, \quad J(u) \leq a \Rightarrow \chi(u,t) = u. \end{array} \tag{34}$$

Define the map $\rho : J^b_{|K\cap N} \to J^a_{|K\cap N}$ by $\rho(u) = \chi(u,1)$. From (34) we obtain that ρ is well defined and it is a retraction. Set

$$\mathcal{J} : J^b_{|K\cap N} \times [0,1] \to J^b_{|K\cap N}, \qquad \mathcal{J}(u,t) = \chi(u,t).$$

The definition of $\mathcal{J}$ implies that, for every $u \in J^b_{|K\cap N}$,

$$\mathcal{J}(u,0) = u \qquad \text{and} \qquad \mathcal{J}(u,1) = \rho(u) \tag{35}$$

and, for any $(u,t) \in J^a_{|K\cap N} \times [0,1]$,

$$\mathcal{J}(u,t) = \mathcal{J}(u,0). \tag{36}$$

From (35) and (36) it follows that $\mathcal{J}$ is $J^a_{|K\cap N}$-homotopic to the identity of $J^a_{|K\cap N}$, that is, $\mathcal{J}$ is a strong deformation retraction, so the pair $(J^b_{|K\cap N}, J^a_{|K\cap N})$ is trivial. Assertions (A_1), (A_2) and Definition 4.8 (ii) show that c_j is not an essential value of $J_{|K\cap N}$. This contradiction concludes our proof. ■

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Chapitre IV
Étude des inégalités hemivariationnelles

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Perturbations of hemivariational inequalities with constraints and applications

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Abstract

The aim of the present paper is to discuss the influence which have certain perturbations on the solution of the eigenvalue problem for hemivariational inequalities on a sphere of given radius. The perturbation results by adding to the hemivariational inequality a term of the type $g^0(x, u(x); v(x))$, where g is a locally Lipschitz nonsmooth and nonconvex energy functional. Applications illustrate the theory.

Introduction

The study of variational inequalities began in the early sixties with the pioneering works of G. Fichera (see [8]), J.L. Lions and G. Stampacchia (see [11]). The connection of this theory with the notion of subdifferential of a convex function was achieved by J.J. Moreau (see [12]), who introduced the notion of convex superpotential.

The mathematical theory of hemivariational inequalities, as well as their applications in Mechanics, Engineering or Economics, were introduced and developed by P.D. Panagiotopoulos (see [20-27]) in the case of nonconvex energy functions. He also defined the notion of nonconvex superpotential (see [19]). An overview of these methods is given in the recent monograph by Z. Naniewicz and P.D. Panagiotopoulos (see [16]). By replacing the subdifferential of a convex function by the generalized gradient (in the sense of F.H. Clarke) of a locally Lipschitz functional, hemivariational inequalities arise whenever the energetic functional associated to a concrete problem is nonconvex. The hemivariational inequalities appear as a generalization of the variational inequalities, but they are much more general than these ones, in the sense that they are not equivalent to minimum problems but, they give rise to substationarity problems. Since one of the main ingredients of this study is based on the notion of Clarke subdifferential of a locally Lipschitz functional, the theory of hemivariational inequalities appears as a new field of Non-smooth Analysis.

Note that all problems formulated in terms of hemivariational inequalities can be formulated "equivalently" as multivalued differential equations. However, the formulation in terms of hemivariational inequalities has a great advantage: that the hemivariational inequalities express a physical principle, the principle of virtual work or power. This fact permits us to use all the advantages of the energetic approach in the mathematical treatement. Moreover, the energetic approach is the only approach towards the development of a solid numerical method.

1 The abstract framework

Let V be a real Hilbert space with the scalar product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$. Assume V is densely and compactly imbedded in $L^p(\Omega;\mathbf{R}^N)$, for some $1 < p < +\infty$ and $N \geq 1$, where Ω is a bounded domain in $\mathbf{R}^m$, $m \geq 1$. In particular, the continuity of this embedding ensures the existence of a positive constant $C_p(\Omega)$ such that

$$\|u\|_{L^p} \leq C_p(\Omega)\|u\|, \qquad \text{for all } u \in V\,.$$

Throughout, the Euclidean norm in $\mathbf{R}^N$ will be denoted by $|\cdot|$, while the duality pairing between $V^\star$ and V (resp., between $(\mathbf{R}^N)^\star$ and $\mathbf{R}^N$) will be denoted by $\langle\cdot,\cdot\rangle_V$ (resp., $\langle\cdot,\cdot\rangle$).

Let $a : V \times V \to \mathbf{R}$ be a continuous, symmetric and bilinear form, which is not necessarily coercive. Let $A : V \to V^\star$ be the self-adjoint bounded linear operator which corresponds to a, that is, for every $u, v \in V$,

$$\langle Au, v\rangle_V = a(u,v)\,.$$

For $r > 0$, set S_r the sphere of radius r in V centered at the origin, i.e.

$$S_r = \{u \in V;\ \|u\| = r\}\,.$$

Consider a mapping $C : S_r \times V \to \mathbf{R}$, to which we impose no continuity assumption. However, for our purpose, a weak kind of compactness hypothesis is given by

(H$_1$) There exists a locally Lipschitz function $f : V \to \mathbf{R}$, even and bounded on S_r, satisfying

$$C(u,v) \geq f^0(u;v), \qquad \text{for all } (u,v) \in S_r \times V\,,\ \text{with } (u,v) = 0\,,$$

and such that the set

$$\{\zeta \in V^\star;\ \zeta \in \partial f(u),\ u \in S_r\}$$

is relatively compact in $V^\star$.

Here $f^0(u;v)$ stands for the Clarke derivative of f at $u \in V$ with respect to the direction $v \in V$, $v \neq 0$, that is

$$f^0(u;v) = \limsup_{\substack{w \to u \\ \lambda \searrow 0}} \frac{f(w + \lambda v) - f(w)}{\lambda}\,.$$

Accordingly, Clarke's generalized gradient $\partial f(u)$ of f at u is defined by

$$\partial f(u) = \{\zeta \in V^\star; f^0(u;v) \geq \langle\zeta, v\rangle_V, \ \text{ for all } v \in V\}\,.$$

Let $j : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ be a Carathéodory function which is locally Lipschitz with respect to the second variable and such that $j(\cdot, 0) \in L^1(\Omega)$. We also assume that this functional satisfies the symmetry condition

(H$_2$) $j(x,y) = j(x,-y)$, for a.e. $x \in \Omega$ and every $y \in \mathbf{R}^N$.

and

(H$_3$) there exist $a_1 \in L^{p/(p-1)}(\Omega)$ and $b \in \mathbf{R}$ such that

$$\mid w \mid \leq a_1(x) + b \mid y \mid^{p-1}, \qquad \text{for a.e. } (x,y) \in \Omega \times \mathbf{R}^N \ \text{ and all } w \in \partial j(x,y)\,.$$

We have denoted by $\partial j(x,y)$ Clarke's generalized gradient of the locally Lipschitz mapping $y \longmapsto j(x,y)$, for some fixed $x \in \Omega$.

Let $\Lambda : V \to V^\star$ be the duality isomorphism

$$(\Lambda u, v)_V = (u,v), \qquad \text{for all } u, v \in V\,.$$

Our last assumption is

(H$_4$) Let $(u_n) \subset S_r$ be an arbitrary sequence which converges weakly in V to some u. Consider a sequence $\zeta_n \in \partial f(u_n)$ such that

$$a(u_n, u_n) + \langle \zeta_n, u_n \rangle_V \to \alpha_0$$

and, for every $w \in L^{p/(p-1)}(\Omega; \mathbf{R}^N)$ verifying

$$w(x) \in \partial j(x, u(x))\,, \qquad \text{for a.e. } x \in \Omega\,,$$

the sequence $\{(A - \lambda_0 \Lambda) u_n\}$ is convergent. Then there exists a strongly convergent subsequence of (u_n) in V. Here λ_0 is defined by

$$\lambda_0 = r^{-2}(\alpha_0 + \int_\Omega \langle w(x), u(x) \rangle dx)\,.$$

In the proof of our main result we shall make use of some notions of Algebraic Topology, for which we refer to [29, Chapter 1] (see also [6,7]). We recall only few basic definitions.

Let X be a metric space and $A \subset X$. A map $r : X \to A$ is said to be a *retraction* if it is continuous, surjective and $r_{|A} = Id$. A retraction r is called to be a *strong deformation retraction* provided there exists a homotopy $F : X \times [0,1] \to X$ of $i \circ r$ and Id_X which satisfies the additional condition $F(x,t) = F(x,0)$, for each $(x,t) \in A \times [0,1]$. Here i stands for the inclusion map of A in X. The metric space X is said to be *weakly locally contractible*, if every point has a neighbourhood which is contractible in X.

Let $\psi : X \to \mathbf{R}$ be a locally Lipschitz functional. For every $a \in \mathbf{R}$, set

$$[\psi \leq a] = \{u \in X; \psi(u) \leq a\}\,.$$

Given $a, b \in \mathbf{R}$ with $a \leq b$, the pair $([\psi \leq b], [\psi \leq a])$ is said to be *trivial* provided that, for every neighbourhoods $[a', a'']$ of a and $[b', b'']$ of b, there exist some closed sets A and B such that $[\psi \leq a'] \subset A \subset [\psi \leq a'']$, $[\psi \leq b'] \subset B \subset [\psi \leq b'']$ and such that A is a strong deformation retract of B.

A real number c is said to be an *essential value* of ψ if, for every $\varepsilon > 0$, there exist $a, b \in (c - \varepsilon, c + \varepsilon)$, with $a < b$ and such that the pair $([\psi \leq b], [\psi \leq a])$ is not trivial. This notion is essentialy due to M. Degiovanni and S. Lancelotti (see [7]).

2 The main result

Let us consider the following eigenvalue hemivariational inequality with constraints:

($\mathbf{P}_1$) Find $(u, \lambda) \in V \times \mathbf{R}$ such that, for all $v \in V$,

$$(1) \qquad \begin{cases} a(u,v) + C(u,v) + \int_\Omega j^0(x, u(x); v(x))dx \geq \lambda(u,v)\,, \\ \|u\| = r\,. \end{cases}$$

Under hypotheses **($\mathbf{H}_1$)-($\mathbf{H}_4$)**, Motreanu and Panagiotopoulos proved in [13, Thm. 4] that this problem admits infinitely many pairs of solutions $(\pm u_n, \lambda_n)$, with all u_n distinct. Moreover, they find the expression of eigenvalues λ_n. Remark that their statement is done under a slight less general hypothesis, namely by assuming a_1=const. in **($\mathbf{H}_3$)**. Examining this proof, we remark that in order to show that the arguments of [13] hold in our case, it is sufficient to verify that the energy functional

$$(1a) \qquad F(u) = \frac{1}{2}a(u,u) + f(u) + J(u), \qquad u \in V\,,$$

is bounded from below on S_r where $J : L^p(\Omega; \mathbf{R}^N) \to \mathbf{R}$ is defined by $J(u) = \int_\Omega j(x, u(x))dx$. Indeed, notice first that, for a.e. $(x,y) \in \Omega \times \mathbf{R}^N$,

$$\begin{aligned} \mid j(x,y) \mid \leq & \mid j(x,0) \mid + \mid j(x,y) - j(x,0) \mid \leq \\ & \mid j(x,0) \mid + \sup\{\mid w \mid; w \in \partial j(x,Y),\ Y \in [0,y]\} \cdot \mid y \mid \leq \\ & \mid j(x,0) \mid + a_1(x) \mid y \mid + b \mid y \mid^p\ . \end{aligned}$$

Therefore

$$\mid J(u) \mid \leq \|j(\cdot,0)\|_{L^1} + \|a_1\|_{L^{p'}} \cdot \|u\|_{L^p} + b\|u\|^p_{L^p}\,.$$

Hence,

$$F_{|S_r}(u) \geq -\frac{1}{2}\|a\| \cdot r^2 - \|f\|_{L^\infty} - \|j(\cdot,0)\|_{L^1} - C_p(\Omega)\|a_1\|_{L^{p'}} r - bC_p^p(\Omega) r^p\,.$$

From now on the proof follows the same lines as in [13].

A natural question arises now: what happens if we perturb (1) in a suitable manner? Perturbation results for the case of equations have been established in [1,2] while perturbation techniques for variational inequalities have been developed in [3,7]. Let us consider the following non-symmetric perturbed hemivariational inequality:

($\mathbf{P}_2$) Find $(u, \lambda) \in V \times \mathbf{R}$ such that

$$(2) \qquad \begin{cases} a(u,v) + C(u,v) + \int_\Omega \Big(j^0(x,u(x);v(x)) + g^0(x,u(x);v(x))\Big)dx + \\ \langle \varphi, v \rangle_V \geq \lambda(u,v)\,, \quad \text{for all } v \in V \\ \|u\| = r\,, \end{cases}$$

where $\varphi \in V^\star$ and $g : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function which is locally Lipschitz with respect to the second variable and such that $g(\cdot, 0) \in L^1(\Omega)$. Fix $\delta > 0$. We make no symmetry assumption on g, but we impose only the growth condition

$(\mathbf{H}_5)$ $\mid w \mid \leq a_2(x) + \delta \mid y \mid^{p-1}$, for a.e. $(x, y) \in \Omega \times \mathbf{R}^N$ and for all $w \in \partial g(x, y)$, where $a_2 \in L^{p/(p-1)}(\Omega)$.

We also assume

$(\mathbf{H}_6)$ The mappings $g(\cdot, 0)$, a_2 and φ satisfy

$$\|a_2\|_{L^{p'}} \leq \delta \qquad \text{and} \qquad \|\varphi\|_{V^\star} \leq \delta\,.$$

As a compactness condition we assume the following variant of $(\mathbf{H}_4)$:

$(\mathbf{H}_7)$ Let $(u_n) \subset S_r$ be an arbitrary sequence which converges weakly in V to some u. Assume $\zeta_n \in \partial f(u_n)$ such that

$$a(u_n, u_n) + \langle \zeta_n, u_n \rangle_V \to \alpha_0$$

and, for every $w, z \in L^{p/(p-1)}(\Omega; \mathbf{R}^N)$ verifying

$$w(x) \in \partial j(x, u(x)) \quad \text{and} \quad z(x) \in \partial g(x, u(x)) \qquad \text{for a.e. } x \in \Omega\,, \tag{2a}$$

the sequence $\{(A - \lambda_0 \Lambda) u_n\}$ is convergent. Then (u_n) is relatively compact in V. Here λ_0 is defined by

$$\lambda_0 = r^{-2}(\alpha_0 + \int_\Omega \langle w(x) + z(x), u(x) \rangle dx)\,.$$

Our aim is to show that the number of solutions of $(\mathbf{P}_2)$ increases as $\delta \to 0$. More precisely, we have

Theorem 1. *Assume hypotheses* $(\mathbf{H}_1)$-$(\mathbf{H}_7)$ *hold.*
Then, for every $n \geq 1$, there exists $\delta_n > 0$ such that, for each $\delta \leq \delta_n$, the problem $(\mathbf{P}_2)$ *admits at least n distinct solutions.*

In the proof of our main result, given in the next section, we shall make use of some techniques from [6,7,13,15].

3 Proof of Theorem 1

We shall follow in the proof a method developed by Degiovanni and Lancelotti in [7].

For every $n \geq 1$, set

$$\Gamma_n = \{S \subset S_r; S \in \mathcal{F}, \gamma(S) \geq n\}\,,$$

where $\mathcal{F}$ denotes the family of closed and symmetric subsets of S_r with respect to the origin and $\gamma(S)$ represents Krasnoselski's genus of the set $S \in \Gamma_n$. Namely, $\gamma(S)$ is the smallest $k \in \mathbf{N} \cup \{+\infty\}$ for which there exists an odd continuous mapping from S into $\mathbf{R}^k \setminus \{0\}$. Motreanu and Panagiotopoulos proved in [13] that the corresponding min-max values of F over Γ_n

$$\beta_n = \inf_{S \in \Gamma_n} \sup_{u \in S} F(u), \qquad n \geq 1\,,$$

are critical values of F on S_r. We first remark that

Lemma 1. *We have that* $\sup_{S_r} F$ *is not achieved and* $\lim_{n\to\infty} \beta_n = \sup_{u\in S_r} F(u)$. *Moreover, there exists a sequence* (b_n) *of essential values of* $F_{|S_r}$ *strictly increasing to* $\sup_{u\in S_r} F(u)$.

Proof. The proof of this result is essentially contained in [7]. It is sufficient to adapt the arguments given in these papers for the case of locally Lipschitz functionals and replacing the classical Fréchet-differentiability by the subdifferentiability in the sense of Clarke. We point out only the main steps of the proof:

i) The functional $F_{|S_r}$ satisfies the Palais-Smale condition (see the proof of Thm. 4 in [13]). So, if there exist $u_0 \in S_r$ and $m < n$ such that $\beta_m = \beta_n \le f(u_0)$, then $\gamma(K_{\beta_m}) \ge n - m + 1$, where

$$K_{\beta_m} = \{u \in S_r;\ F(u) = \beta_m \quad \text{and} \quad \lambda_F(u) = 0\}\,.$$

In the above relation, λ_F is defined by

$$\lambda_F(u) = \min\{\|\xi\|;\ \xi \in \partial F(u)\}\,.$$

It is known (see [4]) that if F is a locally Lipschitz functional then λ_F is lower semi-continuous.

ii) If the sequence (β_n) is stationary and if there exists $u_0 \in S_r$ such that i) holds, then $\gamma(K_{\beta_m}) = +\infty$, for some $m \ge 1$. This is not possible, since S_r is a weakly locally contractible space and K_{β_m} is a compact set, which implies $\gamma(K_{\beta_m}) < +\infty$.

iii) It follows by the previous steps, the definition of Krasnoselski's genus and the fact that $F \not\equiv$const. on S_r, that $\sup_{u\in S_r} F(u)$ is not achieved and $\lim_{n\to\infty} \beta_n = \sup_{u\in S_r} F(u)$. Moreover, without loss of generality, we may assume that $\sup_{u\in S_r} F(u) = +\infty$. Let us define

$$\overline{\Gamma}_n = \{\varphi(S^{n-1});\ \varphi : S^{n-1} \to S_r \text{ is continuous and odd}\},$$

and

$$\overline{\beta}_n = \inf_{C\in\overline{\Gamma}_n} \sup_{u\in C} F(u)\,.$$

Of course, $\overline{\beta}_n \ge \beta_n$, so that $\lim_{n\to\infty} \overline{\beta}_n = \sup_{u\in S_r} F(u) = +\infty$. By Theorem 2.12 of [7] it follows that there exists a sequence (b_n) of essential values of $F_{|S_r}$ strictly increasing to $\sup_{u\in S_r} F(u)$. ■

Notice that the proof of Theorem 4 in [13] works if f is supposed to be only bounded from below on S_r. If $\sup f(u)_{u\in S_r} = \infty$ then $\sup_{u\in S_r} F(u) = \infty$ and $\beta_n \to \infty$, as $n \to \infty$.

We associate to the hemivariational problem $(\mathbf{P}_2)$ the energy function $H : V \to \mathbf{R}$, defined by

$$H(u) = \frac{1}{2}a(u,u) + f(u) + J(u) + G(u) + \langle \varphi, u\rangle_V\,, \tag{2b}$$

where $G(u) = \int_\Omega g(x, u(x))dx$, for every $u \in L^p(\Omega; \mathbf{R}^N)$. The next result asserts that if δ is chosen sufficiently close to 0 in $(\mathbf{H}_5)$ and $(\mathbf{H}_6)$, then H is a small perturbation of the functional F on S_r.

Lemma 2. *For every $\varepsilon > 0$, there exists $\delta_0 > 0$ such that, for all $\delta \leq \delta_0$,*

$$\sup_{u \in S_r} \mid F(u) - H(u) \mid < \varepsilon .$$

Proof. We have

$$\mid g(x,y) \mid \leq \mid g(x,0) \mid + a_2(x) \mid y \mid + \delta \mid y \mid^p .$$

Thus, for all $u \in S_r$,

$$\begin{aligned} \mid F(u) - H(u) \mid \leq & \mid G(u) + \langle \varphi, u \rangle_V \mid \leq \mid G(u) \mid + \delta r \leq \\ & \|g(\cdot,0)\|_{L^1} + \delta C_p(\Omega) r + \delta C_p^p(\Omega) r^p + \delta r < \varepsilon , \end{aligned}$$

for $\delta > 0$ small enough. ■

Lemma 3. *The functional H satisfies the Palais-Smale condition on S_r.*

Proof. Let (u_n) be a sequence in S_r such that $\sup_n \mid H(u_n) \mid < +\infty$ and $\lambda_H(u_n) \to 0$, as $n \to \infty$. The expression of the generalized gradient of H on S_r is given by

$$(3) \qquad \partial(H_{|S_r})(u) = \{\xi - r^{-2}\langle \xi, u \rangle_V \Lambda u ; \xi \in \partial H(u)\} .$$

Consequently, there exists a sequence $(\xi_n) \subset V^\star$ such that

$$(4) \qquad \xi_n \in \partial H(u_n)$$

and

$$(5) \qquad \xi_n - r^{-2}\langle \xi_n, u_n \rangle_V \Lambda u_n \to 0, \qquad \text{strongly in } V^\star .$$

We have to prove that (u_n) is relatively compact.

Using (4), (5) and applying the formula for the generalized gradient of a sum (see, e.g., [5, Prop. 2.3.3]) in the expression of H, one obtains the existence of $\zeta_n \in \partial f(u_n)$, $w_n \in \partial(J_{|V})(u_n)$ and $z_n \in \partial(G_{|V})(u_n)$ such that

$$(6) \quad \Lambda A u_n + \zeta_n + w_n + z_n - r^{-2}\langle \Lambda A u_n + \zeta_n + w_n + z_n, u_n \rangle_V \Lambda u_n + \varphi \to 0 \quad \text{strongly in } V^\star .$$

Moreover, the density of V in $L^p(\Omega; \mathbf{R}^N)$ implies (see [4, Thm. 2.2])

$$\partial(J_{|V})(u) \subset \partial J(u) \qquad \text{and} \qquad \partial(G_{|V})(u) \subset \partial G(u) .$$

It is well known that the embedding $V^\star \subset L^{p/(p-1)}(\Omega; \mathbf{R}^N)$ is compact. Thus one can suppose, passing eventually to subsequences, that

$$(7) \qquad w_n \to w \qquad \text{strongly in } V^\star$$

$$(8) \qquad z_n \to z \qquad \text{strongly in } V^\star .$$

Furthermore, hypothesis $(\mathbf{H}_4)$ implies that (eventually, at a subsequence),

$$\zeta_n \to \zeta \qquad \text{strongly in } V^\star . \tag{9}$$

Since $\|u_n\| = r$, we can also assume that

$$u_n \rightharpoonup u \qquad \text{weakly in } V . \tag{10}$$

Additionally, we can suppose that

$$\{a(u_n, u_n)\} \qquad \text{converges in } \mathbf{R} \tag{11}$$

and

$$\langle w_n + z_n, u_n \rangle_V \to \langle w + z, u \rangle_V . \tag{12}$$

Using the upper semicontinuity of the Clarke generalized gradient (see [5, Prop. 2.1.5]), the relations (6), (12) and the hypothesis $(\mathbf{H}_1)$, we find

$$w \in \partial(J_{|V})(u) \tag{13}$$

$$z \in \partial(G_{|V})(u) \tag{14}$$

$$\zeta \in \partial f(u) . \tag{15}$$

Applying now Theorem 2.7.5 in [5], the relations (13) and (14) yield

$$w(x) \in \partial j(x, u(x)) \qquad \text{for a.e. } x \in \Omega \tag{16}$$

$$z(x) \in \partial g(x, u(x)) \qquad \text{for a.e. } x \in \Omega . \tag{17}$$

Set

$$\lambda_0 = r^{-2}(\alpha_0 + \int_\Omega \langle w(x) + z(x), u(x) \rangle dx) ,$$

where

$$\alpha_0 = \lim_{n\to\infty} \{a(u_n, u_n) + \langle w_n + z_n, u_n \rangle_V\} .$$

Relations (6)-(12) allow us now to deduce that the sequence $\{(A - \lambda_0 \Lambda) u_n\}$ converges strongly in $V^\star$. Then, by $(\mathbf{H}_7)$, there exists a strongly convergent subsequence of (u_n), which concludes our proof. ■

Lemma 4. *If u is a critical point of $H_{|S_r}$ then there exists $\lambda \in \mathbf{R}$ such that (u, λ) is a solution of* $(\mathbf{P}_2)$.

Proof. We have, for every $u \in V$,

$$\partial H(u) = \Lambda A u + \partial(J_{|V})(u) + \partial(G_{|V})(u) + \varphi . \tag{18}$$

Since $0 \in \partial(H_{|S_r})(u)$, it follows by (3) and (14) that there exists

$$w \in \partial(J_{|V})(u) \subset \partial J(u) \qquad \text{and} \qquad z \in \partial(G_{|V})(u) \subset \partial G(u) \tag{19}$$

such that u is a solution of

$$\Lambda Au + w + z + \varphi = r^{-2}\langle \Lambda Au + w + z + \varphi, u\rangle_V \, \Lambda u\,. \tag{20}$$

Moreover (see [5, Thm. 2.7.3]), for every $u \in L^p(\Omega; \mathbf{R}^N)$,

$$\partial J(u) \subset \int_\Omega \partial j(x, u(x))dx \qquad \text{and} \qquad \partial G(u) \subset \int_\Omega \partial g(x, u(x))dx\,.$$

Thus, by (19), the mappings $w, z : \Omega \to (\mathbf{R}^N)^\star$ satisfy

$$w(x) \in \partial j(x, u(x)) \qquad \text{for a.e. } x \in \Omega\,, \tag{21}$$

$$z(x) \in \partial g(x, u(x)) \qquad \text{for a.e. } x \in \Omega\,, \tag{22}$$

and, for all $v \in V$,

$$\langle w, v\rangle_V = \int_\Omega \langle w(x), v(x)\rangle dx\,, \tag{23}$$

$$\langle z, v\rangle_V = \int_\Omega \langle z(x), v(x)\rangle dx\,. \tag{24}$$

Set

$$\lambda = r^{-2}(\langle \Lambda Au + \varphi, u\rangle_V + \int_\Omega \langle w(x) + z(x), u(x)\rangle dx)\,. \tag{25}$$

It follows by (20)-(25) that, for every $v \in V$,

$$\begin{aligned}
&\lambda(u, v) - a(u, v) - \langle \varphi, v\rangle_V = \int_\Omega \langle w(x) + z(x), v(x)\rangle dx \le \\
&\int_\Omega \max\Big\{ \langle \mu, v(x)\rangle; \mu \in \partial(w + z)(x, u(x))\Big\} dx \le \\
&\int_\Omega \max\Big\{ \langle \mu_1, v(x)\rangle; \mu_1 \in \partial w(x, u(x))\Big\} dx + \\
&\int_\Omega \max\Big\{ \langle \mu_2, v(x)\rangle; \mu_2 \in \partial z(x, u(x))\Big\} dx = \\
&\int_\Omega j^0(x, u(x); v(x))dx + \int_\Omega g^0(x, u(x); v(x))dx\,.
\end{aligned} \tag{26}$$

We have used above the classical inclusion (see [5, Prop. 2.3.3])

$$\partial(w + z)(x, u(x)) \subset \partial w(x, u(x)) + \partial z(x, u(x))\,.$$

We point out that the last equality in (26) holds because of Prop. 2.1.2 from [5]. ■

Proof of Theorem 1. Fix $n \geq 1$. Taking into account Lemma 4, it suffices to motivate the existence of some $\delta_n > 0$ such that, for every $\delta \leq \delta_n$, the functional $H_{|S_r}$ has at least n distinct critical values.

By Lemma 1, let (b_n) be a sequence of essential values of $F_{|S_r}$ strictly increasing to $\sup_{u\in S_r} F(u)$. Fix $n \geq 1$ and some $\varepsilon_0 < \dfrac{1}{2} \min_{1\leq j\leq n-1}(b_{j+1} - b_j)$. We apply Theorem 2.6 from [7] to $F_{|S_r}$ and $H_{|S_r}$. Hence, for every $1 \leq j \leq n-1$, there exists $\eta_j > 0$ such that

$$\sup_{u\in S_r} \mid F(u) - H(u) \mid < \eta_j$$

implies the existence of an essential value c_j of $H_{|S_r}$ in $(b_j - \varepsilon_0, b_j + \varepsilon_0)$. We now apply Lemma 2 for $\varepsilon = \min\{\varepsilon_0, \eta_1, \cdots, \eta_{n-1}\}$. This yields the existence of some $\delta_n > 0$ such that

$$\sup_{u\in S_r} \mid F(u) - H(u) \mid < \varepsilon\,,$$

provided $\delta \leq \delta_n$ in $(\mathbf{H}_5)$ and $(\mathbf{H}_6)$. So, we have obtained that the functional $H_{|S_r}$ has at least n distinct essential values $c_1, \cdots, c_n$ in $(-\infty, b_n + \varepsilon)$. It remains to prove that $c_1, \cdots, c_n$ are critical values of $H_{|S_r}$. Arguing by contradiction, let us assume that c_j is not a critical value of $H_{|S_r}$.

Claim. *There exists $\varepsilon > 0$ so that $H_{|S_r}$ has no critical value in $(c_j - \varepsilon, c_j + \varepsilon)$.*

Proof of Claim. Indeed, if not, there is a sequence (d_n) of critical values of $H_{|S_r}$ with $d_n \to c_j$, as $n \to \infty$. Since d_n is a critical value, there exists $u_n \in S_r$ such that

$$H(u_n) = d_n \qquad \text{and} \qquad \lambda_H(u_n) = 0\,.$$

Now we take into account that $(PS)_{c_j}$ holds. Therefore, up to a subsequence, one can suppose that (u_n) converges to some $u \in S_r$, as $n \to \infty$. By the continuity of H and the lower semicontinuity of λ_H, it follows that

$$H(u) = c_j \qquad \text{and} \qquad \lambda_H(u) = 0\,,$$

which contradicts the initial assumption on c_j and concludes the proof of our Claim.

Now we apply the Noncritical Point Theorem (see [6, Thm. 2.15]), which can be also deduced as a consequence of the Deformation Lemma for locally Lipschitz functionals (see [4, Thm. 3.1]). Thus, for some fixed $c_j - \varepsilon < a < b < c_j + \varepsilon$, there exists a continuous map $\eta : S_r \times [0,1] \to S_r$ such that, for each $(u,t) \in S_r \times [0,1]$,

$$\begin{array}{l} \eta(u,0) = u, \qquad H(\eta(u,t)) \leq H(u), \\ H(u) \leq b \Longrightarrow H(\eta(u,1)) \leq a, \qquad H(u) \leq a \Longrightarrow \eta(u,t) = u\,. \end{array}$$

It follows that the map

$$\rho : [H_{|S_r} \leq b] \to [H_{|S_r} \leq b], \qquad \rho(u) = \eta(u,1)$$

is a retraction. Set

$$\mathcal{H} : [H_{|S_r} \leq b] \times [0,1] \to [H_{|S_r} \leq b], \qquad \mathcal{H}(u,t) = \eta(u,t)\,.$$

We observe that, for every $u \in [H_{|S_r} \leq b]$,

$$\mathcal{H}(u,0) = u \qquad \text{and} \qquad \mathcal{H}(u,1) = \rho(u)\,. \tag{27}$$

Moreover, for each $(u,t) \in [H_{|S_r} \leq a] \times [0,1]$,

$$\mathcal{H}(u,t) = \mathcal{H}(u,0)\,. \tag{28}$$

By (27) and (28) it follows that $\mathcal{H}$ is $[H_{|S_r} \leq a]$-homotopic to the identity of $[H_{|S_r} \leq a]$, i.e., $\mathcal{H}$ is a strong deformation retraction. This means that the pair $([H_{|S_r} \leq b], [H_{|S_r} \leq a])$ is trivial. Therefore, c_j is not an essential value of $H_{|S_r}$. This contradiction concludes our proof. ∎

4 A note of the possible Applications

The perturbation results obtained in the previous Sections may have a serious application in the study of the eigenvalue problems for hemivariational inequalities. Suppose, for instance, that we deal with the eigenvalue problem of two adhesively connected v. Kármán plates [28] and that the interface law has a very complicated form (a zig-zag nonmonotone multivalued diagram). Then one can consider the eigenvalue problem for a simplified interface law which results by "smoothing some parts" of the complicated initial law. With respect to the corresponding nonsmooth nonconvex potential energy (1a) this "simplification procedure" means that we have added an additional nonconvex and nonsmooth energy term (cf. eq. (2b)). The simplified interface law results by the "superposition" of the two nonmonotone multivalued relations given in (2a).

Here we deal with systems having a prescribed cost or weight or consumed energy. This is the meaning of the constraint $\|u\| = r$ and therefore we have an eigenvalue problem for hemivariational inequality on a sphere of a given radius.

Theorem 1 of the present paper holds in all cases of the applications given in [13] Sect.3, where we refer the reader for further information.

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A Perturbation Result for a Double Eigenvalue Hemivariational Inequality with Constraints and Applications

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Abstract

In this paper we prove a perturbation result for a new type of eigenvalue problem introduced by D. Motreanu and P.D. Panagiotopoulos in [9]. The perturbation is made in the nonsmooth and nonconvex term of a double eigenvalue problem on a spherlike type manifold considered in [1]. For our aim we use some techniques related to the Lusternik-Schnirelman theory (including Krasnoselski's genus) and results proved in [4], [5] and [24]. We apply these results in the study of some problems arising in Nonsmooth Mechanics.

1 Introduction

The mathematical theory of hemivariational inequalities and their applications in Mechanics, Engineering or Economics, were introduced and developed by P.D. Panagiotopoulos (see [17-23]). This theory may be considered as an extension of the theory of variational inequalities studied by G. Fichera (see [6]), J.L. Lions and G. Stampacchia (see [8]). However, Hemivariational Inequalities are much more general, in the sense that they are not equivalent to minimum problems, but give rise to substationarity problems.

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In this paper we deal with a new type of eigenvalue problem for hemivariational inequalities, called "double eigenvalue problems" which were introduced by D. Motreanu and P.D. Panagiotopoulos (see [9]). In [1] it is proved a multiplicity result concerning the solutions belonging to a spherelike type manifold. Our aim is to study the effect induced by an arbitrary perturbation made in the nonsmooth and nonconvex term of the symmetric hemivariational inequality considered in [1].

2 The abstract framework

Let V be a real Hilbert space, with the scalar product and the associated norm denoted by $(\cdot,\cdot)_V$ and $\|\cdot\|_V$, respectively. We shall suppose that V is densely and compactly embedded in $L^p(\Omega;\mathbf{R}^N)$ for some $p \geq 2$, where $N \geq 1$ and $\Omega \subset \mathbf{R}^m, m \geq 1$, is a smooth, bounded domain. Throughout in this paper, we shall denote by $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle$ the duality products on V and $\mathbf{R}^N$, respectively. Let us denote by $C_p(\Omega)$ the constant of the (continuous, in particular) embedding $V \subset L^p(\Omega;\mathbf{R}^N)$ which means that

$$\|v\|_{L^p} \leq C_p(\Omega)\cdot\|v\|_V, \text{ for all } v \in V.$$

Let $a_1, a_2 : V \times V \to \mathbf{R}$ be two continuous symmetric bilinear forms on V and let $B_1, B_2 : V \to V$ be two bounded self-adjoint linear operators which are coercive in the sense that

$$(B_i v, v)_V \geq b_i \cdot \|v\|_V^2, \text{ for all } v \in V,\ i = 1,2,$$

for some constants $b_1, b_2 > 0$. For fixed positive numbers a, b, r we consider the submanifold $S_r^{a,b}$ of $V \times V$ described as follows

$$S_r^{a,b} = \{(v_1, v_2) \in V \times V \ : \ a(B_1 v_1, v_1)_V + b(B_2 v_2, v_2)_V = r^2\}.$$

We need to consider the tangent space associated to the manifold defined above, which is

$$T_{(u_1,u_2)} S^r_{a,b} := \{(v_1, v_2) \in V \times V \ : \ a(B_1 u_1, v_1)_V + b(B_2 u_2, v_2)_V = 0\}.$$

Let $j : \Omega \times \mathbf{R}^N \to \mathbf{R}$ satisfy the following assumptions

(i) $j(\cdot, y)$ is measurable in Ω for each $y \in \mathbf{R}^N$ and $j(\cdot, 0)$ is essentially bounded in Ω;

(ii) $j(x,\cdot)$ is locally Lipschitz in $\mathbf{R}^N$ for a.e. $x \in \Omega$.

Throughout this paper we shall use the notation j_y^0 for Clarke's generalized directional derivative (see [3]) of j with respect to the second variable y, i.e.,

$$j_y^0(x, y; z) = \limsup_{\substack{w \to y \\ \lambda \downarrow 0}} \frac{j(x, w + \lambda z) - j(x, w)}{\lambda},$$

with $x \in \Omega$, $y, z \in \mathbf{R}^N$ and $\lambda \in \mathbf{R}$. Accordingly, Clarke's generalized gradient $\partial_y j(x, y)$ of the locally Lipschitz map $j(x,\cdot)$ is defined by

$$\partial_y j(x, y) = \{\xi \in \mathbf{R}^N \ : \ \langle \xi, z\rangle \leq j_y^0(x, y; z),\ \forall z \in \mathbf{R}^N\}.$$

As Rădulescu and Panagiotopoulos observed in [24], we may request that j satisfies a slight more general growth condition than the classical one (see the hypothesis (H_1) in [13])

(H_1) There exist $\theta \in L^{\frac{p}{(p-1)}}(\Omega)$ and $\rho \in \mathbf{R}$ such that

$$|z| \leq \theta(x) + \rho|y|^{p-1}, \tag{1}$$

for a.e. $(x, y) \in \Omega \times \mathbf{R}^N$ and each $z \in \partial_y j(x, y)$.

Let us consider a real function $C : S_r^{a,b} \times V \times V \to \mathbf{R}$ to which we impose no continuity assumption. We are now prepared to consider the following double eigenvalue problem : Find $u_1, u_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$(P^1_{r,a,b}) \begin{cases} a_1(u_1, v_1) + a_2(u_2, v_2) + C((u_1, u_2), v_1, v_2) + \\ + \int\limits_\Omega j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x))dx \geq \\ \geq \lambda_1(B_1 u_1, v_1)_V + \lambda_2(B_2 u_2, v_2)_V, \ \forall \ v_1, v_2 \in V, \\ \\ a(B_1 u_1, u_1)_V + b(B_2 u_2, u_2)_V = r^2. \end{cases}$$

We impose the following hypothesis

(H_2) There exist two locally Lipschitz maps $f_i : V \to \mathbf{R}$, bounded on $\pi_i(S_r^{a,b})$, $(i = 1, 2)$ respectively, and such that the following inequality holds

$$C((u_1, u_2), v_1, v_2) \geq f_1^0(u_1; v_1) + f_2^0(u_2; v_2), \tag{2}$$
$$\forall \ (u_1, u_2) \in S_r^{a,b} \text{ and } \forall \ (v_1, v_2) \in T_{(u_1,u_2)} S_r^{a,b},$$

In addition we suppose that the sets

$$\{z \in V^* \ : \ z \in \partial f_i(u_i), u_i \in \pi_i(S_r^{a,b})\}$$

are relatively compact in V^*, for $i = 1, 2$.

Define the map $(A_1, A_2) : V \times V \to V^* \times V^*$ by the relation

$$\langle (A_1, A_2)(u_1, u_2), (v_1, v_2) \rangle_{V \times V} = a_1(u_1, v_1) + a_2(u_2, v_2) \tag{3}$$

and the duality map $\Lambda : V \times V \to V^* \times V^*$ given by

$$\langle \Lambda(u_1, u_2), (v_1, v_2) \rangle_{V \times V} = a(B_1 u_1, v_1)_V + b(B_2 u_2, v_2)_V. \tag{4}$$

We also assume

(H_3) For every sequence $\{(u_n^1, u_n^2)\} \subset S_r^{a,b}$ with $u_n^i \rightharpoonup u_i$ weakly in V, for any $z_n^i \in \partial f_i(u_n^i)$, with

$$a_i(u_n^i, u_n^i) + \langle z_n^i, u_n^i \rangle_V \to \alpha_i \in \mathbf{R}, \tag{5}$$

$i = 1, 2$, and for all $w \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ which satisfies the relation

$$w(x) \in \partial_y j(x, (u_1 - u_2)(x)) \text{ for a.e. } x \in \Omega, \tag{6}$$

such that

$$[(A_1, A_2) - \lambda_0 \cdot \Lambda] (u_n^1, u_n^2)$$

converges in $V^* \times V^*$, where

$$\lambda_0 = r^{-2}(\alpha_1 + \alpha_2 + \int_\Omega \langle w(x), (u_1 - u_2)(x)\rangle\, dx), \tag{7}$$

there exists a convergent subsequence of (u_n^1, u_n^2) in $V \times V$ (thus, in $S_r^{a,b}$).

(H_4) j is even with respect to the second variable, i.e.,

$$j(x, -y) = j(x, y), \text{ for a.e. } x \in \Omega, \text{ and any } y \in \mathbf{R}^N,$$

and f_i is even on $\pi_i(S_r^{a,b})$ i.e.,

$$f_i(-u_i) = f_i(u_i), \text{ for all } (u_1, u_2) \in S_r^{a,b}, i = 1, 2.$$

By assuming the hypotheses $(H_1), (H_2), (H_3)$ and (H_4), it is proved in [1] that the double eigenvalue problem $(P^1_{r,a,b})$ admits infinitely many pairs of solutions $\{\pm(u_n^1, u_n^2), (\lambda_n^1, \lambda_n^2)\} \subset S_r^{a,b} \times \mathbf{R}^2$. Moreover, it is found the expression of the eigenvalues λ_n^1 and λ_n^2. The aim of this paper is to answer a natural question: what happens if we perturb $(P^1_{r,a,b})$ in a suitable manner? For proving our main result we need some notions of Algebraic Topology which may be found in [26]. We recall now only some basic definitions.

Let X be a metric space and $A \subset X$. We said that a map $r : X \to A$ is a *retraction* if it is continuous, surjective and fulfills $r_{|A} = Id$. A retraction r is called to be a *strong deformation retraction* if there exists a homotopy $F : X \times [0,1] \to X$ of $i \circ r$ and Id_X such that $F(x,t) = F(x,0)$, for each $(x,t) \in A \times [0,1]$. Here i stands for the inclusion map of A in X. We call X to be *weakly locally contractible,* if every point has a contractible neighbourhood in X. Let $\xi : X \to \mathbf{R}$ be a locally Lipschitz functional. Set, for every $a \in \mathbf{R}$

$$[\xi \le a] := \{u \in X; \xi(u) \le a\}.$$

Let us fix $a, b \in \mathbf{R}$ with $a \le b$. The pair $([\xi \le b], [\xi \le a])$ is called *trivial* if, for every neighbourhoods $[a', a'']$ of a and $[b', b'']$ of b, there exist some closed sets A and B such that $[\xi \le a'] \subset A \subset [\xi \le a'']$, $[\xi \le b'] \subset B \subset [\xi \le b'']$ and such that A is a strong deformation retract of B.

The next notion is essentialy due to M. Degiovanni and S. Lancelotti (see [5]).

A real number c is said to be an *essential value* of ξ if, for every $\epsilon > 0$, there exist $a, b \in (c - \epsilon, c + \epsilon)$, with $a < b$ and such that the pair $([\xi \le b], [\xi \le a])$ is not trivial.

Let us consider an arbitrary element ϕ in V^* and $g : \Omega \times \mathbf{R}^N \to \mathbf{R}$ a Caratheodory function which is locally Lipschitz with respect to the second variable and such that $g(\cdot, 0) \in L^1(\Omega)$. Let us consider the following non-symmetric perturbed double eigenvalue problem: find $(u_1, u_2) \in V \times V$ and $(\lambda_1, \lambda_2) \in \mathbf{R}^2$ such that

$$(P^2_{r,a,b}) \begin{cases} a_1(u_1, v_1) + a_2(u_2, v_2) + C((u_1, u_2), v_1, v_2) + \\ + \int_\Omega \{j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x)) + \\ g_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x))\} dx + \\ + < \phi, v_1 >_V + < \phi, v_2 >_V \ge \\ \ge \lambda_1 (B_1 u_1, v_1)_V + \lambda_2 (B_2 u_2, v_2)_V, \ \forall\ v_1, v_2 \in V, \\ \\ a(B_1 u_1, u_1)_V + b(B_2 u_2, u_2)_V = r^2. \end{cases}$$

Fix $\delta > 0$. We impose to g the growth condition

(H_5) There exists $\theta_1 \in L^{\frac{p}{(p-1)}}(\Omega)$ such that

$$|z| \leq \theta_1(x) + \delta|y|^{p-1}, \tag{8}$$

for a.e. $(x, y) \in \Omega \times \mathbf{R}^N$ and each $z \in \partial_y g(x, y)$.

Let us denote by J and G the (locally Lipschitz, by hypotheses (H_1) and (H_5)) functionals from $L^p(\Omega; \mathbf{R}^N)$ into $\mathbf{R}$, defined by

$$J(u) = \int\limits_\Omega j(x, u(x))dx \text{ and } G(u) = \int\limits_\Omega g(x, u(x))dx.$$

We associate to the problems $(P^1_{r,a,b})$ and $(P^2_{r,a,b})$ the energy functions $I_1, I_2 : V \times V \to \mathbf{R}$, defined by

$$\begin{aligned} I_1(u_1, u_2) = \frac{1}{2} \cdot [a_1(u_1, u_1) + a_2(u_2, u_2)] + \\ + f_1(u_1) + f_2(u_2) + J(u_1 - u_2), \end{aligned} \tag{9}$$

and

$$I_2(u_1, u_2) = I_1(u_1, u_2) + G(u_1 - u_2) + \langle \phi, u_1 \rangle_V + \langle \phi, u_2 \rangle_V, \tag{10}$$

for all $u_1, u_2 \in V$.

We denote by Υ the family of closed and symmetric with respect to the origin $0_{V \times V}$, subsets of $S^{a,b}_r$. Let us denote, as usually, by $\gamma(S)$ the Krasnoselski's genus of the set $S \in \Upsilon$, that is, the smallest integer $k \in \mathbf{N} \cup \{+\infty\}$ for which there exists an odd continuous mapping from S into $\mathbf{R}^k \backslash \{0\}$. For every $n \geq 1$, set

$$\Gamma_n = \{S \subset S^{a,b}_r \ : \ S \in \Upsilon, \gamma(S) \geq n\}$$

Recall that the corresponding minimax values of I_1 over Γ_n

$$\beta_n = \inf_{S \subset \Gamma_n} \sup_{(u_1, u_2) \in S} \{I_1(u_1, u_2)\},$$

are critical values of I_1 on $S^{a,b}_r$ (see [1, Theorem 1]).

3 Preliminary Results

The first result of this section concerns the functional I_1.

LEMMA 1. *Let* $s := \sup\limits_{(u_1, u_2) \in S^{a,b}_r} \{I_1(u_1, u_2)\}$. *Then the supremum is not achieved and* $\lim\limits_{n \to \infty} \beta_n = s$. *Moreover, there exists a sequence* (b_n) *of essential values of the restriction of* I_1 *at* $S^{a,b}_r$, *strictly increasing to* s.

Proof. This result is essentially proved in [24] (see Lemma 1) by using the ideas of M. Degiovanni and S. Lancelotti (see [5], Theorem 2.12). The only difference is that now, we work not on a sphere but on the Riemannian manifold $S^{a,b}_r$. It is sufficient to point out that this is a weakly locally contractible space as the usual sphere in V is, and the fact that I_1 satisfies the Palais-Smale condition on $S^{a,b}_r$ as was proved in [1]. With these remarks, the proof of the Lemma 1 follows the same steps with the one in [24].

For continuing, we need two aditional assumptions

(H_6) The following inequalities hold

$$\|\theta_1\|_{L^{\frac{p}{p-1}}} \leq \delta,\ \|g(\cdot,0)\|_{L^1} \leq \delta \text{ and } \|\phi\|_{V^*} \leq \delta. \tag{11}$$

The second assumption is actually a variant of the compactness hypothesis (H_3)

(H_7) For every sequence $\{(u_n^1, u_n^2)\} \subset S_r^{a,b}$ with $u_n^i \rightharpoonup u_i$ weakly in V, for any $z_n^i \in \partial f_i(u_n^i)$, with

$$a_i(u_n^i, u_n^i) + \left\langle z_n^i, u_n^i \right\rangle_V + < \phi, u_n^i >_V \to \alpha_i \in \mathbf{R}, \tag{12}$$

$i = 1, 2$ and for all $w, z \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ which satisfies the relations

$$\begin{gathered} w(x) \in \partial_y j(x, (u_1 - u_2)(x)), \\ z(x) \in \partial_y g(x, (u_1 - u_2)(x)), \text{ for a.e. } x \in \Omega, \end{gathered} \tag{13}$$

such that

$$[(A_1, A_2) - \lambda_0 \cdot \Lambda]\,(u_n^1, u_n^2)$$

converges in $V^* \times V^*$, where,

$$\lambda_0 = r^{-2}(\alpha_1 + \alpha_2 + \int\limits_\Omega \langle w(x) + z(x), (u_1 - u_2)(x) \rangle \, dx), \tag{14}$$

there exists a convergent subsequence of (u_n^1, u_n^2) in $V \times V$.

The next result proves that if $\delta > 0$ is sufficiently small in the hypotheses (H_5) and (H_6), then I_2 is a small perturbation of I_1 on $S_r^{a,b}$.

LEMMA 2. *For every $\epsilon > 0$, there exists $\delta_0 > 0$ such that, for all $\delta \leq \delta_0$ we have*

$$\sup_{(u_1,u_2) \in S_r^{a,b}} |I_1(u_1, u_2) - I_2(u_1, u_2)| < \epsilon.$$

Proof. By using mainly the Lebourg's mean value theorem for locally Lipschitz functionals (see [3]) and the hypothesis (H_5) we find

$$|G(u)| \leq \|g(\cdot,0)\|_{L^1} + \|\theta_1\|_{L^{\frac{p}{p-1}}} \cdot \|u\|_{L^p} + \delta \|u\|_{L^p}^p.$$

Taking into account the hypothesis (H_6) and the fact that $(u_1, u_2) \in S_r^{a,b}$ we derive that

$$\begin{gathered} |I_1(u_1, u_2) - I_2(u_1, u_2)| = |G(u_1 - u_2) + < \phi, u_1 >_V + < \phi, u_2 >_V | \leq \\ \leq \|g(\cdot,0)\|_{L^1} + \|\theta_1\|_{L^{\frac{p}{p-1}}} \cdot C_p(\Omega) \cdot r \cdot (\frac{1}{\sqrt{ab_1}} + \frac{1}{\sqrt{bb_2}}) + \\ + \delta \cdot C_p^p(\Omega) \cdot r^p (\frac{1}{\sqrt{ab_1}} + \frac{1}{\sqrt{bb_2}})^p + \delta \cdot r \cdot (\frac{1}{\sqrt{ab_1}} + \frac{1}{\sqrt{bb_2}}) < \epsilon, \end{gathered}$$

for $\delta > 0$ small enough.

LEMMA 3. *The functional I_2 satisfies the Palais-Smale condition on $S_r^{a,b}$.*

Proof. For the beginning it is important to remark that the expression of the generalized gradient $\partial(I_{|S_r^{a,b}})$ at the point $(u_1, u_2) \in S_r^{a,b}$ is given by the formula

$$\partial(I_{|S_r^{a,b}})(u_1, u_2) = \{\xi - r^{-2}\langle \xi, (u_1, u_2)\rangle_{V\times V} \cdot \Lambda(u_1, u_2) \ : \ \xi \in \partial I(u_1, u_2)\},$$

where $\Lambda : V \times V \to V^* \times V^*$ is the appropriate duality map given in (4). Here, the duality $\langle \cdot, \cdot \rangle_{V\times V}$ is taken for the norm

$$\|(u_1, u_2)\|_{V\times V} := \sqrt{a(B_1 u_1, u_1)_V + b(B_2 u_2, u_2)_V}, \ \forall\, u_1, u_2 \in V.$$

Let us consider a sequence $(u_n^1, u_n^2) \subset S_r^{a,b}$ such that

$$\sup_n |(I_{1_{|S_r^{a,b}}})(u_n^1, u_n^2)| < +\infty$$

and such that there exists some sequence $J_n \subset V^* \times V^*$ fulfilling the conditions

$$J_n \in \partial I_1(u_n^1, u_n^2), \quad n \geq 1$$

and

$$J_n - r^{-2}\left\langle J_n, (u_n^1, u_n^2)\right\rangle_{V\times V} \cdot \Lambda(u_n^1, u_n^2) \to 0, \tag{15}$$

strongly in $V^* \times V^*$. For concluding it suffices to prove that $\{(u_n^1, u_n^2)\}$ contains a convergent subsequence in $V \times V$. Under hypothesis (H_1) the functionals J and G are Lipschitz continuous on bounded sets in $L^p(\Omega; \mathbf{R}^N)$ and their generalized gradients satisfy (cf. Clarke [3], Theorem 2.7.5)

$$\partial J(v) \subset \int_\Omega \partial_y j(x, v(x))dx$$

and

$$\partial G(v) \subset \int_\Omega \partial_y g(x, v(x))dx, \ \forall\, v \in L^p(\Omega; \mathbf{R}^N).$$

The density of V into $L^p(\Omega; \mathbf{R}^N)$ allows us to apply Theorem 2.2 of Chang [2]. Thus, we obtain

$$\partial(J_{|V})(v) \subset \partial J(v),$$

and

$$\partial(G_{|V})(v) \subset \partial G(v), \ \forall\, v \in V.$$

From $J_n \in \partial I_1(u_n^1, u_n^2)$ we derive that there exists $z_n^i \in \partial f_i(u_n^i) (i = 1, 2),\ w_n \in \partial(J_{|V})(u_n^1 - u_n^2)$ and $z_n \in \partial(G_{|V})(u_n^1 - u_n^2)$ such that

$$J_n = (a_1(u_n^1, \cdot) + z_n^1 + \phi, a_2(u_n^2, \cdot) + z_n^2 + \phi) + K^*(w_n) + K^*(z_n),$$

where $K : V \times V \to V$ is the map given by

$$K(v_1, v_2) = v_1 - v_2.$$

By the above considerations we have that

$$w_n(x) \in \partial_y j(x, (u_n^1 - u_n^2)(x))$$

and

$$z_n(x) \in \partial_y g(x, (u_n^1 - u_n^2)(x)), \text{ for a.e. } x \in \Omega.$$

By the relation (15) we get

$$\Big(a_1(u_n^1, \cdot) + z_n^1 + \phi, a_2(u_n^2, \cdot) + z_n^2 + \phi\Big) + K^*(w_n) + K^*(z_n) -$$

$$-r^{-2}\langle[(a_1(u_n^1, \cdot) + z_n^1 + \phi, a_2(u_n^2, \cdot) + z_n^2 + \phi) + K^*(w_n) + K^*(z_n)],$$
$$(u_n^1, u_n^2)\rangle_{V\times V} \cdot \Lambda(u_n^1, u_n^2) \to 0, \quad \text{strongly in } V^* \times V^*.$$

Since the sequence (u_n^1, u_n^2) is contained in $S_r^{a,b}$ and by the coercivity property of B_1 and B_2 it follows that each sequence (u_n^1) and (u_n^2) is bounded in V. So, up to a subsequence, we may conclude that

$$u_n^i \rightharpoonup u_i, \text{ weakly in } V, \text{ for some } u_i \in V, (i = 1, 2).$$

The compactness assumptions in the hypothesis (H_2) implies that, again up to a subsequence,

$$z_n^i \to z_i, \quad \text{strongly in } V^*, \text{ for some } z_i \in V^* \ (i = 1, 2).$$

Also we have

$$\begin{aligned} w_n &\in \partial(J_{|V})(u_n^1 - u_n^2) \subset \partial J(u_n^1 - u_n^2), \\ z_n &\in \partial(G_{|V})(u_n^1 - u_n^2) \subset \partial G(u_n^1 - u_n^2). \end{aligned} \tag{16}$$

The compactness of the embedding $V \subset L^p(\Omega; \mathbf{R}^N)$ provides the convergences

$$u_n^i \to u_i, \quad \text{strongly in } L^p(\Omega; \mathbf{R}^N), \ (i = 1, 2). \tag{17}$$

Since J and G are locally Lipschitz on $L^p(\Omega; \mathbf{R}^N)$, the above property ensures that (w_n) and (z_n) are bounded in $L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$. By the reflexivity of $L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ and the compactness of the embedding $L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N) \subset V^*$, there exist $w, z \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ such that, up to a subsequence,

$$w_n \to w \ \text{ strongly in } V^* \ \text{ and weakly in } L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$$

and

$$z_n \to z \ \text{ strongly in } V^* \ \text{ and weakly in } L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N).$$

Proposition 2.1.5 in Clarke [3] and the relations (16) and (17) yield

$$\begin{aligned} w &\in \partial J(u_1 - u_2), \\ z &\in \partial G(u_1 - u_2). \end{aligned} \tag{18}$$

With the above remarks we may suppose that

$$a_i(u_n^i, u_n^i) \ \text{ converges in } \mathbf{R}, \ i = 1, 2,$$

and

$$\left\langle \left[(z_n^1 + \phi, z_n^2 + \phi) + K^*(w_n) + K^*(z_n) \right], (u_n^1, u_n^2) \right\rangle_{V \times V}$$

possesses a convergent subsequence in $\mathbf{R}$. From (15) and taking into account the convergences stated above we derive that

$$\left(a_1(u_n^1, \cdot), a_2(u_n^2, \cdot) \right) - \lambda_0 \cdot \Lambda(u_n^1, u_n^2),$$

converges strongly in $V^* \times V^*$, where λ_0 is the one required in (H_7). So, hypothesis (H_7) allows us to conclude that (u_n^1, u_n^2) has a convergent subsequence in $V \times V$, so in $S_r^{a,b}$. Thus the Palais-Smale condition for the functional I on $S_r^{a,b}$ is satisfied and the proof is now complete.

LEMMA 4. *If $u = (u_1, u_2)$ is a critical point of $I_{2_{|S_r^{a,b}}}$ then there exists a pair $(\lambda_1, \lambda_2) \subset \mathbf{R}^2$ such that $((u_1, u_2), (\lambda_1, \lambda_2))$ is a solution of the problem $(P^2_{r,a,b})$.*

Proof. Since u is a critical point for $I_{2_{|S_r^{a,b}}}$, it follows that

$$0_{V \times V} \in \left(\partial I_{2_{|S_r^{a,b}}} \right) (u_1, u_2) \tag{19}$$

Taking into account the expression of the generalized gradient of the restriction of I_2 at $S_r^{a,b}$, we may conclude the existence of an element $\xi \in \partial I_2(u_1, u_2)$ such that

$$\xi - r^{-2} \langle \xi, (u_1, u_2) \rangle_{V \times V} \cdot \Lambda(u_1, u_2) = 0 \tag{20}$$

By the Clarke's calculus and the inclusions stated in the proof of Lemma 3 we derive

$$\partial I_2(u_1, u_2)(v_1, v_2) \subset a_1(u_1, v_1) + a_2(u_2, v_2) +$$

$$+ \partial f_1(u_1)v_1 + \partial f_2(u_2)v_2 + \int_\Omega \partial_y j(x, (u_1 - u_2)(x))(v_1 - v_2)(x)dx +$$

$$+ \int_\Omega \partial_y g(x, (u_1 - u_2)(x))(v_1 - v_2)(x)dx + < \phi, v_1 >_V + < \phi, v_2 >_V,$$

for all $v_1, v_2 \in V$. So, there exists some $z_i \in \partial f_i(u_i)$ $(i = 1, 2)$ and $w, z \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ with

$$w(x) \in \partial_y j(x, (u_1 - u_2)(x)) \quad \text{for a.e. } x \in \Omega,$$

and

$$z(x) \in \partial_y g(x, (u_1 - u_2)(x)) \quad \text{for a.e. } x \in \Omega,$$

such that

$$\langle \xi, (v_1, v_2) \rangle_{V \times V} = a_1(u_1, v_1) + a_2(u_2, v_2) + < z_1, v_1 >_V + < z_2, v_2 >_V +$$

$$+ \int_\Omega < w(x), (v_1 - v_2)(x) > dx + \int_\Omega < z(x), (v_1 - v_2)(x) > dx +$$

$$+ < \phi, v_1 >_V + < \phi, v_2 >_V .$$

From (20) it follows that

$$a_1(u_1,v_1)+a_2(u_2,v_2)+<z_1,v_1>_V+<z_2,v_2>_V$$

$$+\int_\Omega <w(x),(v_1-v_2)(x)>dx+\int_\Omega <z(x),(v_1-v_2)(x)>dx+$$

$$+<\phi,v_1>_V+<\phi,v_2>_V-$$

$$-r^{-2}[a_1(u_1,u_1)+a_2(u_2,u_2)+<z_1,u_1>_V+<z_2,u_2>_V+$$

$$+\int_\Omega <w(x),(u_1-u_2)(x)>dx+\int_\Omega <z(x),(u_1-u_2)(x)>dx+$$

$$+<\phi,u_1>_V+<\phi,u_2>_V]\cdot(a(B_1u_1,v_1)_V+b(B_2u_2,v_2)_V)=0,$$

for all $v_1,v_2\in V$. Set

$$\lambda=r^{-2}[a_1(u_1,u_1)+a_2(u_2,u_2)+<z_1,u_1>_V+<z_2,u_2>_V+$$

$$+\int_\Omega <(w+z)(x),(u_1-u_2)(x)>dx+<\phi,u_1>_V+<\phi,u_2>_V].$$

Let us now observe that we have

$$\int_\Omega \langle (w+z)(x),(v_1-v_2)(x)\rangle dx\le$$

$$\le\int_\Omega \max\{\langle\mu_1,(v_1-v_2)(x)\rangle;\mu_1\in\partial_y j(x,(u_1-u_2)(x))\}+$$

$$+\int_\Omega \max\{\langle\mu_2,(v_1-v_2)(x)\rangle;\mu_2\in\partial_y g(x,(u_1-u_2)(x))\}=$$

$$=\int_\Omega j_y^0(x,(u_1-u_2)(x);(v_1-v_2)(x))dx+$$

$$\int_\Omega g_y^0(x,(u_1-u_2)(x);(v_1-v_2)(x))dx.$$

In the above relation, the last equality holds because of Proposition 2.1.2 in [3]. Taking into account the choice of $z_i(i=1,2),z$ and w, it is easily to observe that if we denote $\lambda_1=\lambda a$ and $\lambda_2=\lambda b$, our hypothesis (H_2) and some simple calculation lead us to the desired conclusion claimed in the formulation of Lemma 4.

4 The main result

With the preliminary results stated in Section 3 we are now prepared to prove our perturbation result.

THEOREM 1. *Assume that the hypotheses $(H_1)-(H_7)$ are fulfilled. Then, for every $n\ge 1$, there exists $\delta_n>0$ such that, for each $\delta\le\delta_n$, the problem $(P^2_{r,a,b})$ admits at least n distinct solutions.*

Proof. Fix $n \geq 1$. By Lemma 4 it suffices to prove the existence of a $\delta_n > 0$ such that, for every $\delta \leq \delta_n$, the functional $I_{2_{|S_r^{a,b}}}$ has at least n distinct critical values. We may use now the conclusion of Lemma 1 and this implies that it is possible to consider a sequence (b_n) of essential values of $I_{1_{|S_r^{a,b}}}$, strictly increasing to s. Choose an arbitrary $\epsilon_0 < \frac{1}{2} \min\limits_{1 \leq i \leq n-1} (b_{i+1} - b_i)$. We now apply Theorem 2.6 from [5] to the functionals $I_{1_{|S_r^{a,b}}}$ and $I_{2_{|S_r^{a,b}}}$. Thus, for every $1 \leq i \leq n-1$, there exists $\eta_i > 0$ such that the relation

$$\sup_{(u_1,u_2) \in S_r^{a,b}} |I_1(u_1,u_2) - I_2(u_1,u_2)| < \eta_i$$

implies the existence of an essential value c_i of $I_{2_{|S_r^{a,b}}}$ in $(b_i - \epsilon_0, b_i + \epsilon_0)$. By taking $\epsilon = \min\{\epsilon_0, \eta_1, \cdots, \eta_{n-1}\}$ in Lemma 2, we derive the existence of a $\delta_n > 0$ such that

$$\sup_{(u_1,u_2) \in S_r^{a,b}} |I_1(u_1,u_2) - I_2(u_1,u_2)| < \epsilon,$$

provided $\delta \leq \delta_n$ in (H_5) and (H_6). So, the functional $I_{2_{|S_r^{a,b}}}$ has at least n distinct essential values $c_1, c_2, \cdots, c_n$ in $(-\infty, b_n + \epsilon)$. For concluding our proof it suffices to show that $c_1 \cdots, c_n$ are critical values of $I_{2_{|S_r^{a,b}}}$. The first step is to prove that there exists $\epsilon > 0$ such that $I_{2_{|S_r^{a,b}}}$ has no critical value in $(c_i - \epsilon, c_i + \epsilon)$. Indeed, if this is not the case, there exists a sequence (d_n) of critical values of $I_{2_{|S_r^{a,b}}}$ with $d_n \to c_i$ as $n \to \infty$. The fact that d_n are critical values for the restriction of I_2 at $S_r^{a,b}$ implies that for every $n \geq 1$, there exists $(u_n^1, u_n^2) \in S_r^{a,b}$ such that

$$I_2(u_n^1, u_n^2) = d_n \text{ and } \lambda^*(u_n^1, u_n^2) = 0,$$

where λ^* is the lower semicontinuous functional defined by

$$\lambda^*(u_1,u_2) := \min\{\|(\xi_1,\xi_2)\|_{V^* \times V^*}; (\xi_1,\xi_2) \in \partial I_2(u_1,u_2)\}.$$

Thus, passing eventually to a subsequence, $(u_n^1, u_n^2) \to (u_1,u_2) \in S_r^{a,b}$, strongly in $V \times V$. The continuity of I_2 and the lower semicontinuity of λ^* implies that

$$I_2(u_1,u_2) = c_i \text{ and } \lambda^*(u_1,u_2) = 0,$$

which contradicts the initial conditions on c_i. Let us fix $c_i - \epsilon < a < b < c_i + \epsilon$. By Lemma 3, I_2 satisfies the Palais-Smale condition on $S_r^{a,b}$. So, for every point $e \in [a,b]$, $(PS)_e$ holds. We have fulfilled the set of conditions which allow us to apply the "Noncritical Interval Theorem" due to J.- N. Corvellec, M. Degiovanni and M. Marzocchi (see Theorem 2.15 in [4]), on the complete metric space $\left(S_r^{a,b}, d(\cdot,\cdot)\right)$, where by $d(\cdot,\cdot)$ we have denoted the geodesic distance on $S_r^{a,b}$, that is, for every points $x, y \in S_r^{a,b}$, $d(x,y)$ is equal to the infimum of the lengths of all paths on $S_r^{a,b}$ joining x and y. We obtain that there exists a continuous map $\eta : S_r^{a,b} \times [0,1] \to S_r^{a,b}$ such that, for each $(u = (u_1,u_2), t) \in S_r^{a,b} \times [0,1]$, are satisfied the conditions

(a) $\eta(u,0) = u$,

(b) $I_2(\eta(u,t)) \leq I_2(u)$,

(c) $I_2(u) \leq b \Longrightarrow I_2(\eta(u,1)) \leq a$,

(d) $I_2(u) \leq a \Longrightarrow \eta(u,t) = u$.

By the above conditions, it follows that the map

$$[I_{2_{|S_r^{a,b}}} \leq b] \ni u \mapsto \eta(u,1) \in [I_{2_{|S_r^{a,b}}} \leq b]$$

is a retraction. Let us define the map $\Psi : [I_{2_{|S_r^{a,b}}} \leq b] \times [0,1] \to [I_{2_{|S_r^{a,b}}} \leq b]$ by the relation

$$\Psi(u,t) = \eta(u,t).$$

Since for every $u \in [I_{2_{|S_r^{a,b}}} \leq b]$, we have

$$\Psi(u,0) = u, \;\; \Psi(u,1) = \eta(u,1),$$

and for each $(u,t) \in [I_{2_{|S_r^{a,b}}} \leq b] \times [0,1]$, the equality $\Psi(u,t) = \Psi(u,0)$ holds, it follows that Ψ is $[I_{2_{|S_r^{a,b}}} \leq b]-$ homotopic to the identity of $[I_{2_{|S_r^{a,b}}} \leq b]$. Thus, Ψ is a strong deformation retraction which implies that the pair

$$\left([I_{2_{|S_r^{a,b}}} \leq b], [I_{2_{|S_r^{a,b}}} \leq a]\right)$$

is trivial. With this argument, we get that c_i is not an essential value of the restriction of I_2 at $S_r^{a,b}$. This is the contradiction which concludes our proof.

5 Applications

In many problems arising in Mechanics and Engineering the cost or the weight of the structure may be expressed as a linear function of the norm of the unknown function. Thus the constraint that we have imposed $\|u\|_V = r($ or, equivalently, $a\|u_1\|^2 + b\|u_2\|^2 = r^2)$ means that we have a system with prescribed cost or weight, or in some cases energy consumption. The stability analysis of such a system involving nonconvex nonsmooth potential functions (called also nonconvex superpotential) leads to the treatment of a double eigenvalue problem for hemivariational inequalities on a spherelike manifold. We begin with two mathematical examples and then we shall give some applications from Mechanics.

5.1 Perturbations of a coupled semilinear Poisson equation

First, we consider the case of the problem $(P^1_{r,a,b})$ in which $C \equiv 0, \; B_1 = B_2 = id_V, \; a = b = 1$. Moreover a_1, a_2 are coercive, in the sense that

$$a_i(v,v) \geq \bar{a}_i \|v\|_V^2, \;\; \forall v \in V, i = 1,2,$$

for some constants $\bar{a}_1, \bar{a}_2 > 0$ and $j : \mathbf{R} \to \mathbf{R}$ is the primitive

$$j(t) = \int_0^t \varphi(\tau) d\tau, \ t \in \mathbf{R},$$

with $\varphi : \mathbf{R} \to \mathbf{R}$ even, locally bounded, measurable and satisfying the subcritical growth condition : for some $1 \leq p < \frac{2m}{m-2}$, if $m \geq 3$ ($1 \leq p < +\infty$, if $m = 1, 2$), we have

$$|\varphi(t)| \leq c_1 + c_2 |t|^{p-1}, \ \forall t \in \mathbf{R}.$$

It is known that

$$\partial j(t) \subset [\underline{\varphi}(t), \overline{\varphi}(t)], \ \forall t \in \mathbf{R},$$

where

$$\underline{\varphi}(t) = \lim_{\delta \to 0} \operatorname{essinf} \{\varphi(s) \ ; \ |t - s| < \delta\}$$

and

$$\overline{\varphi}(t) = \lim_{\delta \to 0} \operatorname{esssup} \{\varphi(s) \ ; \ |t - s| < \delta\}$$

(see [2]). Suppose further the sign condition of Chang [2]

$$\underline{\varphi}(t) > 0 \text{ if } t < 0 \text{ and } \overline{\varphi}(t) < 0 \text{ if } t > 0.$$

Let us consider that the superpotential j gives rise to a very irregular graph $[\xi, \partial j(\xi)]$ (i.e. the graph of ∂j has many zig - zag etc.). Then we consider the eigenvalue problem $(P^2_{r,a,b})$, where g^0_y is appropriately chosen in order to "smoother a little bit" the graph $[\xi, \partial j(\xi)]$, i.e. the graph $[\xi, \partial j(\xi) + \partial g(\xi)]$ has a smaller number of irregularities than the graph $[\xi, \partial j(\xi)]$. In the present case we may consider that

$$\partial j(t) + \partial g(t) \subset [\underline{\varphi}(t) + \underline{\varphi}^1(t), \overline{\varphi}(t) + \overline{\varphi^1}(t)], \ \forall t \in \mathbf{R}.$$

In fact, we consider

$$g(t) = \int_0^t \varphi^1(\tau) d\tau, \ t \in \mathbf{R},$$

where $\varphi^1 : \mathbf{R} \to \mathbf{R}$ is locally bounded, measurable and satisfies the subcritical growth condition

$$|\varphi^1(t)| \leq c_3 + c_4 |t|^{p-1}, \ \forall t \in \mathbf{R}$$

Note that we do not need to impose to φ^1 that it is even, as we have assumed on φ. Obviously, Theorem 1 applies on every sphere $\|v_1\|^2_V + \|v_2\|^2_V = r^2$ of $V \times V$, with a sufficiently small $r > 0$. More precisely, for every $n \geq 1$, there exists $\delta_n > 0$ such that if c_3 and c_4 are chosen smaller than δ_n, then the perturbed problem $(P^2_{r,a,b})$ admits at least n distinct solutions.

As a specific example of application of Theorem 1, we consider the coupled semilinear Poisson equations on a bounded domain Ω in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$ in the double eigenvalue problem

$$\triangle u_1 + \lambda_1 u_1 \in [\underline{\varphi}(u_1(x) - u_2(x)), \overline{\varphi}(u_1(x) - u_2(x))] \text{ for a.e. } x \in \Omega$$

$$\triangle u_2 + \lambda_2 u_2 \in [-\overline{\varphi}(u_1(x) - u_2(x)), -\underline{\varphi}(u_1(x) - u_2(x))] \text{ for a.e. } x \in \Omega$$
$$u_1 = u_2 = 0 \text{ on } \partial\Omega.$$

Here $\lambda_1, \lambda_2 \in \mathbf{R}$ are the eigenvalues, u_1, u_2 are the corresponding eigenfunctions and $\underline{\varphi}, \overline{\varphi}$ are determined above for the function $\varphi : \mathbf{R} \to \mathbf{R}$. We choose $V = H_0^1(\Omega)$,

$$a_1(u,v) = a_2(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \ \forall u, v \in H_0^1(\Omega),$$

$$(B_1 u, v)_{H_0^1} = (B_2 u, v)_{H_0^1} = \int_\Omega u \cdot v dx, \ \forall u, v \in H_0^1(\Omega),$$

$j : \mathbf{R} \to \mathbf{R}$ being equal to the primitive of φ as we considered above and, for simplicity, $C \equiv 0$. Notice that each eigensolution of the hemivariational inequality appearing in the problem $(P^1_{r,a,b})$ represents a weak solution of the Dirichlet system above. Under the growth condition for φ as above and the assumptions from the section 2 on j, Theorem 1 in [1] implies the existence of infinitely many double eigenfunctions $(u_n^1, u_n^2) \in S_r^{a,b}$, with $u_n^1, u_n^2 \in H_0^1(\Omega) \cap H^2(\Omega)$ for the foregoing Dirichlet problem.

Further, we consider the perturbed eigenvalue problem

$$\triangle u_1 + \lambda_1 u_1 \in [\underline{\varphi}(u_1(x) - u_2(x)) + \underline{\varphi^1}(u_1(x) - u_2(x)),$$
$$\overline{\varphi}(u_1(x) - u_2(x)) + \overline{\varphi^1}(u_1(x) - u_2(x))] \text{ for a.e. } x \in \Omega$$
$$\triangle u_2 + \lambda_2 u_2 \in [-\overline{\varphi}(u_1(x) - u_2(x)) + \underline{\varphi^1}(u_1(x) - u_2(x)),$$
$$-\underline{\varphi}(u_1(x) - u_2(x)) + \overline{\varphi^1}(u_1(x) - u_2(x))] \text{ for a.e. } x \in \Omega$$
$$u_1 = u_2 = 0 \text{ on } \partial\Omega,$$

where φ^1 is chosen as in the previous example and satisfies the conditions therein. Then, our Theorem 1 applies and we obtain that the perturbed Dirichlet problem considered above admits infinitely many distinct solutions. Notice that c_3 and c_4 must be sufficiently small, in the same sense as in the first case considered in this section.

5.2 Adhesively connected von Kármán plates. Buckling for given cost or weight.

In the framework of the theory of elastic von Kármán plates, i.e. of plates having large deflections, we consider two or more such plates connected with an adhesive material. The behaviour of the adhesive material may be described by a relation of the form

$$-f \in \partial j(u_1 - u_2), \tag{21}$$

(cf. [22], p. 109).The graph of $\{f, u_1 - u_2\}$ may be a zig-zag graph with complete vertical branches in the most general case. Concerning the derivation and study of the corresponding hemivariational inequalities we refer to [16], [22]. We assume that we have two plates Ω_1 and $\Omega_2, \Omega_i \subset \mathbf{R}^2, i = 1, 2$, which are adhesively connected on $\Omega' \subset \Omega_i, i = 1, 2$. The plates have the boundaries Γ_1 and Γ_2 respectively and $\overline{\Omega} \cap \Gamma_i = \emptyset, i = 1, 2$. The boundaries are assumed to be Lipschitzian and are not subjected to any loading on Ω_1 and Ω_2 vertical to the middle plate plane or parallel to it. We assume that $\Omega_1 \equiv \Omega_2$ as subsets of $\mathbf{R}^2$ and we denote both Ω_1 and Ω_2 by Ω. The plates are only subjected along their boundaries Γ_1 and Γ_2 to continuously distributed compressive forces, i.e.

$$\sigma_{\alpha\beta i} n_{\alpha_i} = \lambda_i g_{\alpha_i} \quad \alpha, \beta = 1, 2, \; i = 1, 2,$$

where $\sigma = \{\sigma_{\alpha\beta}\}$ denotes the stress tensor for the in-plane action of the plate, $n = \{n_\alpha\}$ is the outher unit normal vector to Γ_1 or to $\Gamma_2, g_i = \{g_{1_i}, g_{2_i}\}$ is a given force distribution, which is self equilibrated, i.e. for each plate

$$\int\limits_{\Gamma_i} g_{\alpha_i} ds = 0, \quad \int\limits_{\Gamma_i} (x_1 g_{2_i} - g_{1_i} x_2) ds = 0, \quad i = 1, 2.$$

Here $\lambda_i, i = 1, 2,$ is a real number which measures the magnitude of the compressive forces having the direction $g_i, i = 1, 2,$ along the boundaries of the plates. These compressive forces may cause buckling of the composite plate with partial debonding of the adhesive material. As in [15], p. 455 and in [21] p. 234, where the analogous buckling problem for variational inequalities is formulated, the notion of "reduced variational solution" is introduced and we obtain the following eigenvalue problem: Find $u_1, u_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$a_1(u_1, v_1) + a_2(u_2, v_2) + (C_1(u_1), v_1)_V + (C_2(u_2), v_2)_V +$$

$$\int\limits_{\Omega} j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x)) dx \geq \lambda_1 (B_1 u_1, v_1)_V + \lambda_2 (B_2 u_2, v_2)_V,$$

for all $v_1, v_2 \in V$. Here V is the real Sobolev space $H^2(\Omega)$ with inner product $(\cdot, \cdot)_V$, $a_i(u_i, v_i)$ is the bending energy of the plate i, $(C_i(u_i), v_i)$, with $C_i(\cdot)$ a nonlinear compact operator, is the bending energy of the plate i due to the stretching of the same plate, $j^0(x, u_1 - u_2; v_1 - v_2)$ denotes the directional derivative in the sense of Clarke at the state $(u_1 - u_2)(x)$ and in the direction $(v_1 - v_2)(x)$ at x, and $(B_i u_i, v_i)$ is given by the relation (7.2.13) of [21], i.e.

$$(B_i u_i, v_i) = -\int\limits_{\Omega_i} h_i \sigma^0_{\alpha\beta_i} u_{i,\alpha} v_{i,\beta} dx \; \forall \; u_i, v_i \in V,$$

for $i = 1, 2$. Here h_i denotes the thickness of the plate i and $\alpha\beta_i^0$ the stress field in the plane of the plate i caused by the forces $g_{\alpha_i} (\alpha, \beta = 1, 2, \; i = 1, 2)$. Moreover we note that on Γ_i, concerning the plate bending, boundary conditions which guarantee the coercivity of the bilinear forms $a_i(\cdot, \cdot), i = 1, 2$, are assumed to hold. For instance the built-in boundary conditions $u_i = \frac{\partial u_i}{\partial n} = 0, i = 1, 2$, or the simple support boundary conditions $u_i = 0, M_i(u_i) = 0, i = 1, 2$, where M_i denotes the bending element of the $i-$th plate. Further we shall not need for the operators B_i the property that $(B_i u_i, v_i) > 0 \, \forall \, u_i \in V, u_i \neq 0$, as it is the case in the corresponding theory (see [15]) of eigenvalue problems for variational inequalities but the stronger property of coercivity (this property is a consequence of the assumption that the stress vector on the boundary of each subdomain Ω_{0_i} of $\Omega, i = 1, 2$, is directed outside of Ω_{0_i}, i.e. that each subdomain of the plate is subjected to compressive forces, (cf. [15] p. 457)). Further we express the total cost or weight of the structure by the form $\sum_{i=1}^{2} a_i \left(B_i u_i, v_i\right) = r^2$, where a_i are given positive constants. We get that for the arising double eigenvalue problem for hemivariational inequalities $(P^1_{r,a,b})$ the hypotheses are satisfied and the multiplicity result of Theorem 1 in [1] holds.

5.2.1 Perturbations of the buckling problem of a sandwich beam of prescribed weight.

Let us now consider the perturbed hemivariational inequality : Find $u_1, u_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$a_1(u_1, v_1) + a_2(u_2, v_2) + (C_1(u_1), v_1)_V + (C_2(u_2), v_2)_V +$$

$$\int_\Omega j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x))dx \geq \lambda_1 \left(B_1 u_1, v_1\right)_V + \lambda_2 \left(B_2 u_2, v_2\right)_V,$$

for all $v_1, v_2 \in V$. One can assume that the graph $[\xi, \partial j(\xi) + \partial g(\xi)]$ is much more regular than the graph $[\xi, \partial j(\xi)]$. Further one can assume that the graph $[\xi, \partial j(\xi) + \partial g(\xi)]$ is monotone, a fact which in the framework of a numerical calculation is beneficial. Moreover, in the monotone case one can consider the corresponding variational inequality - eigenvalue problem and get some useful comparison results (especially in the case of simple eigenvalue problems for which there exist certain results for variational inequalities (see [7]).

5.2.2 Fuzzy effects superimposed on an adhesive contact law.

Let us put ourselves in the framework of the previous example of adhesively connected plates and let us consider the following interface law (see Panagiotopoulos [22], p. 77)

$$-f(x) \in \partial j([u](x)) + \partial g(u(x)), \tag{22}$$

where ∂g describes the fuzzy effects. We recall that g results in the following manner (see [25])

Let l be an open subset of the real line $\mathbf{R}$ and let M be a measurable subset of l such that for every open and nonempty subset I of l, $\mathrm{mes}(I \cap (l - M))$ is > 0. Let

$$r(u(x)) = \begin{cases} +b_1 & \text{if} \quad u(x) \in M \\ -b_1 & \text{if} \quad u(x) \notin M \end{cases}$$

and $g(u) = \int_0^u r(u^*)du^*$. Then g is Lipschitzian and

$$\partial g(u) = [-b_2, b_1], \ \forall u(x) \in l.$$

Thus $\partial g(u(\cdot))$ has an infinite number of jumps in l where each jump is identified with the interval $[-b_2, b_1]$. In the composite law (22), the zero of this interval lies on the graph of $[\xi, j(\xi)]$ and the zone $[-b_2, b_1]$ around this graph describes the fuzzy nature of the adhesive contact law. Note that existence results related to fuzzy effects have been studied by Naniewicz and Panagiotopoulos in [14] p. 132. Here we can apply our results to the perturbed problem $(P^2_{r,a,b})$, i.e. to the system related to the interface law (22). Our Lemma 2 shows that if the fuzzy effect tends to disappear then the energy of the perturbed problem tends to the energy of the initial nonfuzzy problem. On the other hand, by Theorem 1, the number of solutions of the perturbed problem tends to infinity if the perturbation given by the fuzzy effect tends to zero. We also remark that our results hold if the fuzzy effect is linked to a subcritical growth, but is arbitrary, in the sense that it has no symmetry.

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EXISTENCE THEOREMS OF HARTMAN-STAMPACCHIA TYPE FOR HEMIVARIATIONAL INEQUALITIES AND APPLICATIONS

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Abstract

We give some versions of theorems of Hartman-Stampacchia's type for the case of Hemivariational Inequalities on compact or on closed and convex subsets in infinite and finite dimensional Banach spaces. Several problems from Nonsmooth Mechanics are solved with these abstract results.

Keywords and phrases: Hemivariational inequalities, Clarke subdifferential, Monotone operator, Set valued mappings.

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1 Introduction and the main results

The well-known theorem of Hartman-Stampacchia (see [3], Lemma 3.1, or [5], Theorem I.3.1) asserts that if V is a finite dimensional Banach space, $K \subset V$ is compact and convex, $A : K \to V^\star$ is continuous, then there exists $u \in K$ such that, for every $v \in K$,

$$\langle Au, v - u \rangle \geq 0 . \tag{1}$$

If we weak the hypotheses and consider the case where K is a closed and convex subset of the finite dimensional space V, Hartman and Stampacchia proved (see [5], Theorem I.4.2) that a necessary and sufficient condition which ensures the existence of a solution to Problem (1) is that there is some $R > 0$ such that a solution u of (1) with $\|u\| \leq R$ satisfies $\|u\| < R$.

The purpose of this paper is to extend these classical results in the framework of Hemivariational Inequalities. These inequalities appear as a generalization of Variational Inequalities, but they are much more general than these ones, in the sense that they are not equivalent to minimum problems but give rise to substationarity problems. The mathematical theory of Hemivariational Inequalities has been developed by P.D. Panagiotopoulos, as well as their applications in Mechanics, Engineering or Economics (see the monographs [6], [8], [9] and the references cited therein for a treatment of this theory and further comments).

Let V be a real Banach space and let $T : V \to L^p(\Omega, \mathbf{R}^k)$ be a linear and continuous operator, where $1 \leq p < \infty$, $k \geq 1$, and Ω is a bounded open set in $\mathbf{R}^N$. Throughout this paper, K is a subset of V, $A : K \to V^*$ an operator and $j = j(x,y) : \Omega \times \mathbf{R}^k \to \mathbf{R}$ is a Carathéodory function which is locally Lipschitz with respect to the second variable $y \in \mathbf{R}^k$ and satisfies the following assumption

(j) there exists $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbf{R})$ and $h_2 \in L^\infty(\Omega, \mathbf{R})$ such that

$$|z| \leq h_1(x) + h_2(x)|y|^{p-1},$$

for a.e. $x \in \Omega$, every $y \in \mathbf{R}^k$ and $z \in \partial j(x,y)$. Denoting by $Tu = \hat{u}$, $u \in V$, our aim is to study the problem

(P) Find $u \in K$ such that, for every $v \in K$,

$$\langle Au, v - u \rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0.$$

We have denoted by $j^0(x,y;h)$ the (partial) Clarke derivative of the locally Lipschitz mapping $j(x,\cdot)$ at the point $y \in \mathbf{R}^k$ with respect to the direction $h \in \mathbf{R}^k$, where $x \in \Omega$, and by $\partial j(x,y)$ the Clarke generalized gradient of this mapping at $y \in \mathbf{R}^k$, that is

$$j^0(x,y;h) = \limsup_{\substack{y' \to y \\ t \downarrow 0}} \frac{j(x, y' + th) - j(x,y')}{t};$$

$$\partial j(x,y) = \{z \in \mathbf{R}^k : \ \langle z, h \rangle \leq j^0(x,y;h), \quad \text{for all } h \in \mathbf{R}^k\}$$

The euclidean norm in $\mathbf{R}^k$, $k \geq 1$, resp. the duality pairing between a Banach space and its dual will be denoted by $|\cdot|$, resp. $\langle \cdot\, , \cdot \rangle$. We also denote by $\|\cdot\|_p$ the norm in the space $L^p(\Omega, \mathbf{R}^k)$ defined by

$$\|\hat{u}\|_p = \left(\int_\Omega |\, \hat{u}(x)\, |^p \, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

In order to state our existence results for the problem (P), we need the following definitions.

Definition 1. The operator $A : K \to V^*$ is w^*-demicontinuous if for any sequence $\{u_n\} \subset K$ converging to u, the sequence $\{Au_n\}$ converges to Au for the w^*-topology in V^*.

Definition 2. The operator $A : K \to V^*$ is continuous on finite dimensional subspaces of K if for any finite dimensional space $F \subset V$, which intersects K, the operator $A|_{K \cap F}$ is demicontinuous, that is $\{Au_n\}$ converges weakly to Au in V^* for each sequence $\{u_n\} \subset K \cap F$ which converges[1] to u.

Remark 1. In reflexive Banach spaces the following hold:
a) the w^*-demicontinuity and demicontinuity are the same.
b) a demicontinuous operator $A : K \to V^*$ is continuous on finite dimensional subspaces of K.

The following result is a generalized form of the Hartman-Stampacchia Theorem for the case of Hemivariational Inequalities in infinite dimensional real Banach spaces; namely it generalizes Theorem 6 in [13] and Theorem 2.1 in [14] for the framework of such inequalities.

Theorem 1. Let K be a compact and convex subset of the infinite dimensional Banach space V and let j satisfy the condition (j). If the operator $A : K \to V^*$ is w^*-demicontinuous, then the problem (P) admits a solution.

In finite dimensional Banach spaces the above theorem has the following equivalent form.

Corollary 1. Let V be a finite dimensional Banach space and let K be a compact and convex subset of V. If the assumption (j) is fulfilled and if $A : K \to V^*$ is a continuous operator, then the problem (P) has at least a solution.

In Section 2 the proof of Theorem 1 will be based on Corollary 1; for this reason Corollary 1 will be proved before this theorem.

Remark 2. The condition of w^*-demicontinuity on the operator $A : K \to V^*$ in Theorem 1 may be replaced equivalently by the assumption:

(A_1) the mapping $K \ni u \to \langle Au, v\rangle$ is weakly upper semi-continuous, for each $v \in V$.

Indeed, since on the compact set K the weak-topology is in fact the normed topology, we can replace equivalently the weak upper semi-continuity by upper semi-continuity. So we have to prove that the w^*-demicontinuity of A follows from the assumption (A_1); but for any sequence $\{u_n\} \subset K$ converging to u one finds (by (A_1)):

$$\limsup_{n\to\infty} \langle Au_n, v\rangle \le \langle Au, v\rangle$$

and

$$\limsup_{n\to\infty} \langle Au_n, -v\rangle \le \langle Au, -v\rangle \Longleftrightarrow \liminf_{n\to\infty} \langle Au_n, v\rangle \ge \langle Au, v\rangle,$$

for each fixed point $v \in V$. Thus, there exists $\lim_{n\to\infty}\langle Au_n, v\rangle$, and

$$\lim_{n\to\infty} \langle Au_n, v\rangle = \langle Au, v\rangle,$$

for every $v \in V$. Consequently, the sequence $\{Au_n\}$ converges to Au for the w^*-topology in V^*.

[1] By " converges" we always mean "strongly (or norm) converges"

Remark 3. If A is w^*-demicontinuous, $\{u_n\} \subset K$ and $u_n \to u$, then

$$\lim_{n\to\infty} \langle Au_n, u_n \rangle = \langle Au, u \rangle.$$

This follows from the w^*-boundedness of $\{Au_n\}$ in V^* (as a w^*-convergent sequence) and from the fact that in real dual Banach spaces each w^*-bounded set is a (strongly) bounded set[2] (see [12], Prop. IV.5.2). Thus, in this case, one can write

$$\lim_{n\to\infty} \langle Au_n, v - u_n \rangle = \langle Au, v - u \rangle, \tag{2}$$

for each $v \in V$.

This last fact will be helpful in the proof of the theorems in Section 2.

Weaking more the hypotheses on K by assuming that K is a closed, bounded and convex subset of the Banach space V, we need some more about the operators A and T (see Theorem 2). We first recall that an operator $A : K \to V^*$ is said to be monotone if, for every $u, v \in K$,

$$\langle Au - Av, u - v \rangle \geq 0.$$

Thus we can formulate the following result, which is the corresponding variant for Hemivariational Inequalities of Theorem 1.1 in [3].

Theorem 2. Let V be a reflexive infinite dimensional Banach space and let $T : V \to L^p(\Omega, \mathbf{R}^k)$ be a linear and compact operator. Assume K is a closed, bounded and convex subset of V and $A : K \to V^*$ is monotone and continuous on finite dimensional subspaces of K. If j satisfies the condition (j) then the problem (P) has at least one solution.

We also give a generalization of Theorem III.1.7. in [5] by

Theorem 3. Assume that the same hypotheses as in Theorem 2 hold without the assumption of boundedness of K. Then a necessary and sufficient condition for the hemivariational inequality (P) to have a solution is that there exists $R > 0$ with the property that at least one solution of the problem

$$\begin{cases} u_R \in K \cap \{u \in V; \|u\| \leq R\}; \\ \langle Au_R, v - u_R \rangle + \displaystyle\int_\Omega j^0(x, \hat{u}_R(x); \hat{v}(x) - \hat{u}_R(x))dx \geq 0, \\ \text{for every } v \in K \text{ with } \|v\| \leq R, \end{cases} \tag{3}$$

satisfies the inequality $\|u_R\| < R$.

A basic tool in our proofs will be the following auxiliary result.

[2]This generally holds true in the topological dual of a real Hausdorff barreled locally convex space.

Lemma 1. (a) If it is satisfied the assumption (j) and V_1, V_2 are nonempty subsets of V, then the mapping $V_1 \times V_2 \to \mathbf{R}$ defined by:

$$(u,v) \to \int_\Omega j^0(x, \hat{u}(x), \hat{v}(x))dx \tag{4}$$

is upper semi-continuous.

(b) Moreover, if $T : V \to L^p(\Omega, \mathbf{R}^k)$ is a linear compact operator, then the above mapping is weakly upper semi-continuous.

Proof. a) Let $\{(u_m, v_m)\}_{m\in\mathbb{N}} \subset V_1 \times V_2$ be a sequence converging to $(u,v) \in V_1 \times V_2$, as $m \to \infty$. Since $T : V \to L^p(\Omega, \mathbf{R}^k)$ is continuous, it follows that

$$\hat{u}_m \to \hat{u}, \quad \hat{v}_m \to \hat{v} \ \text{ in } \ L^p(\Omega, \mathbf{R}^k), \quad \text{as } \ m \to \infty$$

There exists a subsequence $\{(\hat{u}_n, \hat{v}_n)\}$ of the sequence $\{(\hat{u}_m, \hat{v}_m)\}$ such that

$$\limsup_{m\to\infty} \int_\Omega j^0(x, \hat{u}_m(x); \hat{v}_m(x))dx = \lim_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x))dx.$$

By Proposition 4.11 in [4], one may suppose the existence of two functions $\hat{u}_0, \hat{v}_0 \in L^p(\Omega, \mathbf{R}^+)$, and of two subsequences of $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ denoted again by the same symbols and such that:

$$|\hat{u}_n(x)| \le \hat{u}_0(x), \qquad |\hat{v}_n(x)| \le \hat{v}_0(x),$$

$$\hat{u}_n(x) \to \hat{u}(x), \qquad \hat{v}_n(x) \to \hat{v}(x), \quad \text{as } \ n \to \infty$$

for a.e. $x \in \Omega$. On the other hand, for each x where holds true the condition (j) and for each $y, h \in \mathbf{R}^k$, there exists $z \in \partial j(x,y)$ such that

$$j^0(x,y;h) = \langle z, h\rangle = \max\{\langle w, h\rangle : \ w \in \partial j(x,y)\},$$

(see [1], Prop 2.1.2). Now, by (j),

$$|j^0(x,y;h)| \le |z| \ |h| \le (h_1(x) + h_2(x)|y|^{p-1}) \cdot |h| \ .$$

Consequently, denoting $F(x) = (h_1(x) + h_2(x)|\hat{u}_0(x)|^{p-1})|\hat{v}_0(x)|$, we find that

$$|j^0(x, \hat{u}_n(x); \hat{v}_n(x))| \le F(x),$$

for all $n \in \mathbb{N}$ and for a.e. $x \in \Omega$.

From Holder's Inequality and from the condition (j) for the functions h_1 and h_2 it follows that $F \in L^1(\Omega, \mathbf{R})$. Fatou's Lemma yields

$$\lim_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x))dx \le \int_\Omega \limsup_{n\to\infty} j^0(x, \hat{u}_n(x); \hat{v}_n(x))dx.$$

Next, by the upper-semicontinuity of the mapping $j^0(x, .; .)$ (see [1], Prop. 2.1.1) we get that

$$\limsup_{n\to\infty} j^0(x, \hat{u}_n(x); \hat{v}_n(x)) \le j^0(x, \hat{u}(x); \hat{v}(x))\,,$$

for a.e. $x \in \Omega$, because

$$\hat{u}_n(x) \to \hat{u}(x) \quad \text{and} \quad \hat{v}_n(x) \to \hat{v}(x), \quad \text{as } n \to \infty$$

for a.e. $x \in \Omega$. Hence

$$\limsup_{m\to\infty} \int_\Omega j^0(x, \hat{u}_m(x); \hat{v}_m(x))dx \le \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x))dx,$$

which proves the upper-semicontinuity of the mapping defined by (4).

b) Let $\{(u_m, v_m)\}_{m\in\mathbb{N}} \subset V_1 \times V_2$ be now a sequence weakly-converging to $\{u, v\} \in V_1 \times V_2$, as $m \to \infty$. Thus $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ weakly as $m \to \infty$. Since $T : V \to L^p(\Omega, \mathbf{R}^k)$ is a linear compact operator, it follows that

$$\hat{u}_m \to \hat{u}, \quad \hat{v}_m \to \hat{v} \ \text{ in } \ L^p(\Omega, \mathbf{R}^k).$$

From now on the proof follows the same proof as in the case a). ∎

2 Proof of the theorems

2.1 Proof of Corollary 1

Arguing by contradiction, for every $u \in K$, there is some $v = v_u \in K$ such that

$$\langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx < 0.$$

For every $v \in K$, set

$$N(v) = \{u \in K; \langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx < 0\}.$$

For any fixed $v \in K$ the mapping $K \to \mathbf{R}$ defined by

$$u \longmapsto \langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx$$

is upper semi-continuous, by Lemma 1 and the continuity of A. Thus, by the definition of the upper semi-continuity, $N(v)$ is an open set. Our initial assumption implies that $\{N(v); v \in K\}$ is a covering of K. Hence, by the compactness of K, there exist $v_1, \cdots, v_n \in K$ such that

$$K \subset \bigcup_{j=1}^n N(v_j).$$

Let $\rho_j(u)$ be the distance from u to $K \setminus N(v_j)$. Then ρ_j is a Lipschitz map which vanishes outside $N(v_j)$ and the functionals

$$\psi_j(u) = \frac{\rho_j(u)}{\sum_{i=1}^n \rho_i(u)}$$

define a partition of the unity subordinated to the covering $\{\rho_1, \cdots, \rho_n\}$. Moreover, the mapping $p(u) = \sum_{j=1}^{n} \psi_j(u) v_j$ is continuous and maps K into itself, because of the convexity of K. Thus, by Brouwer's fixed point Theorem, there exists u_0 in the convex closed hull of $\{v_1, \cdots, v_n\}$ such that $p(u_0) = u_0$. Define

$$q(u) = \langle Au, p(u) - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \widehat{p(u)}(x) - \hat{u}(x))dx.$$

The convexity of the map $j^0(\hat{u}; \cdot)$ (see [1], Lemma 1) and the fact that $\sum_{j=1}^{n} \psi_j(u) = 1$ imply

$$q(u) \leq \sum_{j=1}^{n} \psi_j(u) \langle Au, v_j - u\rangle + \sum_{j=1}^{n} \psi_j(u) \int_\Omega j^0(x, \hat{u}(x); \hat{v}_j(x) - \hat{u}(x))dx.$$

For arbitrary $u \in K$, there are only two possibilities: if $u \notin N(v_i)$, then $\psi_i(u) = 0$. On the other hand, for all $1 \leq j \leq n$ (there exists at least such an indice) such that $u \in N(v_j)$, we have $\psi_j(u) > 0$. Thus, by the definition of $N(v_j)$,

$$q(u) < 0, \quad \text{for every } u \in K.$$

But $q(u_0) = 0$, which gives a contradiction. ■

2.2 Proof of Theorem 1

For this proof we need Lemma 2 below. Let F be an arbitrary finite dimensional subspace of V which intersects K. Let $i_{K\cap F}$ be the canonical injection of $K \cap F$ into K and i_F^* be the adjoint of the canonical injection i_F of F into V. Then:

Lemma 2. The operator

$$B : K \cap F \to F^*, \qquad B = i_F^* A i_{K\cap F}$$

is continuous.

Proof. We have to prove that the sequence $\{Bu_n\}$ converges to Bu in F^* for any sequence $\{u_n\} \subset K \cap F$ converging to u in $K \cap F$ (or in V). In order to do this, we prove that the sequence $\{Bu_n\}$ is weakly ($= w^*$) convergent to Bu, because F^* is a finite dimensional Banach space. Let V be an arbitrary point of F; then by the w^*-demicontinuity of the operator $A : K \to V^*$ it follows that

$$\langle Bu_n, v\rangle = \langle i_F^* A i_{K\cap F} u_n, v\rangle = \langle i_F^* A u_n, v\rangle =$$
$$= \langle Au_n \cdot i_F, v\rangle = \langle Au_n, v\rangle \underset{n\to\infty}{\longrightarrow} \langle Au, v\rangle = \langle Bu, v\rangle$$

Therefore $\{Bu_n\}$ converges weakly to Bu. ■

Remark 4. The above lemma also holds true if the operator A is continuous on finite dimensional subspaces of K.

Proof of Theorem 1. For any $v \in K$, set

$$S(v) = \{u \in K; \langle Au, v-u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0\}.$$

Step 1. $S(v)$ is closed set.

We first observe that $S(v) \neq \emptyset$, since $v \in S(v)$. Let $\{u_n\} \subset S(v)$ be an arbitrary sequence which converges to u as $n \to \infty$. We have to prove that $u \in S(v)$. But, by (2), $u_n \in S(v)$ and by the part (a) of Lemma 1, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} [\langle Au_n, v-u_n\rangle + \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))]dx = \\ &= \lim_{n\to\infty} \langle Au_n, v-u_n\rangle + \limsup_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \leq \\ &\leq \langle Au, v-u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx. \end{aligned}$$

This is equivalent to $u \in S(v)$.

Step 2. The family $\{S(v); v \in K\}$ has the finite intersection property.

Let $\{v_1, \cdots, v_n\}$ be an arbitrary finite subset of K and let F be the linear space spanned by this family. Applying Corollary 1 to the operator B defined in Lemma 2, we find $u \in K \cap F$ such that $u \in \cap_{j=1}^n S(v_j)$, which means that the family of closed sets $\{S(v); v \in K\}$ has the finite intersection property. But the set K is compact. Hence

$$\bigcap_{v\in K} S(v) \neq \emptyset,$$

which means that the problem (P) has at least one solution. ■

2.3 Proof of Theorem 2

Let F be an arbitrary finite dimensional subspace of V, which intersects K. Consider the canonical injections $i_{K\cap F} : K \cap F \to K$ and $i_F : F \to V$ and let $i_F^* : V^* \to F^*$ be the adjoint of i_F. Applying Corollary 1 to the continuous operator $B = i_F^\star A i_{K\cap F}$ (see Remark 4) we find some u_F in the compact set $K \cap F$ such that, for every $v \in K \cap F$,

$$(4) \qquad \langle i_F^\star A i_{K\cap F} u_F, v-u_F\rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))dx \geq 0.$$

But

$$(6) \qquad 0 \leq \langle Av - Au_F, v-u_F\rangle = \langle Av, v-u_F\rangle - \langle Au_F, v-u_F\rangle.$$

Hence, by (5), (6) and the observation that $\langle i_F^\star A i_{K\cap F} u_F, v-u_F\rangle =$ $\langle Au_F, v-u_F\rangle$, we have

$$(7) \qquad \langle Av, v-u_F\rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))dx \geq 0,$$

for any $v \in K \cap F$. The set K is weakly closed as a closed convex set; thus it is weakly compact because it is bounded and V is a reflexive Banach-space.

Now, for every $v \in K$ define

$$M(v) = \{u \in K; \langle Av, v-u \rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0\}.$$

The set $M(v)$ is weakly closed by the part (b) of Lemma 1 and by the fact that this set is weakly sequentially dense (see, e.g., [2], pp. 145-149 or [10], p.3). We now show that the set $M = \cap_{v \in K} M(v) \subset K$ is non-empty. To prove this, it suffices to prove that

$$\bigcap_{j=1}^{n} M(v_j) \neq \emptyset, \tag{8}$$

for any $v_1, \cdots, v_n \in K$. Let F be the finite dimensional linear space spanned by $\{v_1, \cdots, v_n\}$. Hence, by (7), there exists $u_F \in F$ such that, for every $v \in K \cap F$,

$$\langle Av, v - u_F \rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))dx \geq 0.$$

This means that $u_F \in M(v_j)$, for every $1 \leq j \leq n$, which implies (8). Consequently, it follows that $M \neq \emptyset$. Therefore there is some $u \in K$ such that, for every $v \in K$,

$$\langle Av, v-u \rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0. \tag{9}$$

We shall prove that from (9) we can conclude that u is a solution of Problem (P). Fix $w \in K$ and $\lambda \in (0, 1)$. Putting $v = (1-\lambda)u + \lambda w \in K$ in (9) we find

$$\langle A((1-\lambda)u + \lambda w), \lambda(w-u) \rangle + \int_\Omega j^0(x, \hat{u}(x); \lambda(\hat{w} - \hat{u})(x))dx \geq 0. \tag{10}$$

But $j^0(x, \hat{u}; \lambda \hat{v}) = \lambda\, j^0(x, \hat{u}; \hat{v})$, for any $\lambda > 0$. Therefore (10) may be written, equivalently,

$$\langle A((1-\lambda)u + \lambda w), w-u \rangle + \int_\Omega j^0(x, \hat{u}(x); (\hat{w} - \hat{u})(x))dx \geq 0. \tag{11}$$

Let F be the vector space spanned by u and w. Taking into account the demi-continuity of the operator $A_{|K \cap F}$ and passing to the limit in (11) as $\lambda \longrightarrow 0$, we obtain that u is a solution of Problem (P). ■

Remark 5. As the set $K \cap \{x \in V;\ \|u\| \leq R\}$ is a closed bounded and convex set in V, it follows from Theorem 2 that the problem (3) in the formulation of our Theorem 3 has at least one solution for any fixed $R > 0$.

2.4 Proof of Theorem 3

The necessity is evident.

Let us now suppose that there exists a solution u_R of (3) with $\|u_R\| < R$. We prove that u_R is solution of (P). For any fixed $v \in K$, we choose $\varepsilon > 0$ small enough so that $w = u_R + \varepsilon(v - u_R)$ satisfies $\|w\| < R$. Hence, by (3),

$$\langle Au_R, \varepsilon(v - u_R)\rangle + \int_\Omega j^0(x, \hat{u}_R(x); \varepsilon(\hat{v} - \hat{u}_R)(x))dx \geq 0$$

and, using again the positive homogeneity of the map $v \longmapsto j^0(u; v)$, the conclusion follows. ■

3 Applications

3.1 Noncoercive Hemivariational Inequalities

We consider noncoercive forms of the coercive and semicoercive hemivariational problems treated in [6], pp. 65-77. The results are more general from the point of view of the absence of the coercivity or the semicoercivity assumption, but less general from the point of view of the boundedness of the set K. For this purpose, let us assume that V is a real Hilbert space and that the continuous injections

$$V \subset [L^2(\Omega, \mathbf{R}^k)]^N \subset V^\star$$

hold, where $V^\star$ denotes the dual space of V. Moreover let $T : V \to L^2(\Omega, \mathbf{R}^k)$, $T(u) = \hat{u}$, $\hat{u}(x) \in \mathbf{R}^k$ be a linear and continuous mapping. Consider the operator A appearing in our abstract framework as $Au = A_1 u + f$, where $f \in V^\star$ is a prescribed element, while A_1 satisfies, respectively, the assumptions of Theorems 1, 2 or 3. Then the theorem 1 holds for the problem

(P_1) Find $u \in K$ such that, for every $v \in K$,

$$\langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0.$$

Moreover, if T is a linear compact operator, then Theorems 2 and 3 also hold for the above problem.

Suppose further that Γ is the Lipschitz boundary of Ω and that the linear mapping $T : V \to L^2(\Gamma, \mathbf{R}^k)$ is continuous. Then the theorem 1 holds for the problem

(P_2) Find $u \in K$ such that, for every $v \in K$,

$$\langle Au, v - u\rangle + \int_\Gamma j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0.$$

Furthermore, if T is compact, then Theorems 2 and 3 remain valid for (P_2).

3.2 Nonmonotone Laws in Networks with Convex Constraints

We shall give now an application in Economics concerning a network flow problem. We follow the basic ideas of W. Prager [7], [11] and for the consideration of the nonlinearities we combine them with the notion of nonconvex superpotential. We refer to [6], p. 191 for the derivation of the formulas.

The generally nonmonotone nonlinearity is caused by the law relating the two branch variables of the network, the "flow intensity" and the "price differential" which here can also be vectors. The problem is formulated as a hemivariational inequality and the existence of its solution is discussed further. We consider networks with directed branches. The nodes are denoted by Latin and the branches by Greek letters. We suppose that we have m nodes and ν branches. We take as branch variables the "flow intensity" s_γ and the "price differential" e_γ. As node variables the "amount of flow" p_k and the "shadow price" u_k are considered. The terminology has been taken from [11]. Moreover each branch may have an "initial price differential" vector e^0_γ. The above given quantities are assembled in vectors e, e^0, u, s, p. The node-branch incidence matrix G is denoted by G, where the lines of G are linearly independent. Upper index T denotes the transpose of a matrix or a vector. The network law is a relation between the "flow intensity" s_γ and the "price differential" e_γ. We accept that s_γ is a nonmonotone function of the e_γ expressed by the relation

$$e_\gamma - e^0_\gamma \in \partial j_\gamma(s_\gamma) + \frac{1}{2}\partial s^T_\gamma C_\gamma s_\gamma\,, \tag{12}$$

where k_γ is a positive definite symmetric matrix and ∂ is the generalized gradient. The graph of the $s_\gamma - e_\gamma$ law is called γ-characteristic.

The problem to be solved consists in the determination for the whole network of the vectors s, e, u, for given vectors p and e_0.

Further let $C = \operatorname{diag}[C_1, \cdots, C_\gamma, \cdots]$ and let the summation $\sum_\gamma$ be extended over all branches. Now we consider the graph which corresponds to the network and a corresponding tree. The tree results from the initial graph by cuting all the branches creating the closed loops. Let us denote by s_T (resp. s_M) the part of the vector s corresponding to the tree branches (resp. to the cut branches giving rise to closed loops). Then we may write instead of $Gs = p$ the relation

$$G_T s_T + G_M s_M = p\,.$$

Here G_T is nonsingular and thus we may write that

$$s = \begin{bmatrix} s_T \\ s_M \end{bmatrix} = \begin{bmatrix} G_T^{-1} \\ 0 \end{bmatrix} p + \begin{bmatrix} -G_T^{-1}G_M \\ I \end{bmatrix} s_M = s_0 + Bs_M\,, \tag{13}$$

where I denotes the unit matrix. Using (12) and (13) we obtain (cf. [6]) a hemivariational inequality with respect to s_M which reads: find $s_M \in \mathbf{R}^{n_1}$ (n_1 is the dimension of s_M) such that

$$\begin{aligned}\sum_\gamma \; & j^0_\gamma((s_0 + Bs_M)_\gamma, (Bs^\star_M - Bs_M)_\gamma) + s^T_M B^T CB(s^\star_M - s_M) + \\ & s^T_0 CB(s^\star_M - s_M) + e^{0T} B(s^\star_M - s_M) \geq 0 \quad \forall s^\star_M \in \mathbf{R}^{n_1}\,.\end{aligned} \tag{14}$$

Let us now assume that the flow intensities s_M are constrained to belong to a bounded and closed convex subset $K \subset \mathbf{R}^{n_1}$ (box constraints are very common). Thus the problem takes the form: find $s_M \in K$ which satisfies (14), for every $s_M^\star \in K$.

Since the rank of B is equal to the number of its columns and C is symmetric and positive definite the same happens for $B^T CB$. In the finite dimensional case treated here, one can easily verify that Corollary 1 holds, if $j_\gamma(\cdot,\cdot)$ satisfies the condition (j). Thus (14) has at least one solution.

3.3 On the Nonconvex Semipermeability Problem

Let us put ourselves within the framework of [6], p. 185, where we have studied nonconvex semipermeability problems. We consider an open, bounded, connected subset Ω of $\mathbf{R}^3$ referred to a fixed Cartesian coordinate system $0x_1x_2x_3$ and we formulate the equation

$$-\Delta u = f \qquad \text{in } \Omega \tag{15}$$

for stationary problems.

Here u represents the temperature in the case of heat conduction problems, whereas in problems of hydraulics and electrostatics the pressure and the electric potential are represented, respectively. We denote further by Γ the boundary of Ω and we assume that Γ is sufficiently smooth ($C^{1,1}$-boundary is sufficient). If $n = \{n_i\}$ denotes the outward unit normal to Γ then $\partial u/\partial n$ is the flux of heat, fluid or electricity through Γ for the aforementioned classes of problems.

We may consider the interior and the boundary semipermeability problems.

In the first class of problems the classical boundary conditions

$$u = 0 \qquad \text{on } \Gamma \tag{16}$$

are assumed to hold, whereas in the second class the boundary conditions are defined as a relation between $\partial u/\partial n$ and u. In the first class the semipermeability conditions are obtained by assuming that $f = \bar{f} + \bar{\bar{f}}$ where $\bar{\bar{f}}$ is given and $\bar{f}$ is a known function of u. Here, we consider (16) for the sake of simplicity. All these problems may be put in the following general framework. For the first class we seek a function u such as to satisfy (15), (16) with

$$f = \bar{f} + \bar{\bar{f}}, \qquad -\bar{f} \in \partial j_1(x,u) \ \text{ in } \Omega\,. \tag{17}$$

For the second class we seek a function u such that (15) is satisfied together with the boundary condition

$$-\frac{\partial u}{\partial n} \in \partial j_2(x,u) \quad \text{on } \Gamma_1 \subset \Gamma \qquad \text{and} \qquad u = 0 \ \text{ on } \Gamma \setminus \Gamma_1\,. \tag{18}$$

Both $j_1(x,\cdot)$ and $j_2(x,\cdot)$ are locally Lipschitz functions and ∂ denotes the generalized gradient. Note, that if $q = \{q_i\}$ denotes the heat flux vector and $k > 0$ is the coefficient of thermal conductivity of the material we may write by Fourier's law that $q_i n_i = -k\partial u/\partial n$.

Let us introduce the notations

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$$

and

$$(f, u) = \int_\Omega f u dx\,.$$

We may ask in addition that u is constrained to belong to a convex bounded closed set $K \subset V$ due to some technical reasons, e.g. constraints for the temperature or the pressure of the fluid etc.

The hemivariational inequalities correspond to the two classes of problems. Let for the first class $V = H_0^1(\Omega)$ and $\bar{\bar{f}} \in L^2(\Omega)$; for the second class $V = \{v : v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma \setminus \Gamma_1\}$ and $f \in L^2(\Omega)$. Then from the Green-Gauss theorem applied to (15), with the definition of (17) and (18) we are led to the following two hemivariational inequalities for the first and for the second class of semipermeability problems respectively

(i) Find $u \in K$ such that

$$a(u, v-u) + \int_\Omega j_1^0(x, u(x); v(x) - u(x))dx \geq (\bar{\bar{f}}, v-u) \qquad \forall v \in K\,. \tag{19}$$

(ii) Find $u \in K$ such that

$$a(u, v-u) + \int_{\Gamma_1} j_2^0(x, u(x); v(x) - u(x))d\Gamma \geq (f, v-u) \qquad \forall v \in K\,. \tag{20}$$

Since $a(\cdot, \cdot)$ is (strongly) monotone on V both in (i) and (ii) and the embeddings $V \subset L^2(\Omega)$ and $V \subset L^2(\Gamma_1)$ are compact we can prove the existence of solutions of (i) and of (ii) by applying Theorem 2 if j_1 and j_2 satisfy the condition (j).

3.4 Adhesively Supported Elastic Plate between two Rigid Supports

Let us consider a Kirchoff plate. The elastic plate is referred to a right-handed orthogonal Cartesian coordinate system $Ox_1x_2x_3$. The plate has constant thickness h_1, and the middle surface of the plate coincides with the Ox_1x_2-plane. Let Ω be an open, bounded and connected subset of $\mathbf{R}^2$ and suppose that the boundary Γ is Lipschitzian ($C^{0,1}$-boundary). The domain Ω is occupied by the plate in its undeformed state. On $\Omega' \subset \Omega$ (Ω' is such that $\overline{\Omega}' \cap \Gamma = \emptyset$) the plate is bonded to a support through an adhesive material. We denote by $\zeta(x)$ the deflection of the point $x = (x_1, x_2, x_3)$ and by $f = (0, 0, f_3)$, $f_3 = f_3(x)$ (hereafter called f for simplicity) the distributed load of the considered plate per unit area of the middle surface. Concerning the laws for adhesive forces and the formulation of the problems we refer to [9]. Here we make the additional assumption that the displacements of the plate are prevented by some rigid supports. Thus we may put as an additional assumption the following one:

$$z \in K, \tag{21}$$

where K is a convex closed bounded subset of the displacement space. One could have e.g. that $a_0 \leq z \leq b_0$ etc.

We assume that any type of boundary conditions may hold on Γ. Here we assume that the plate boundary is free. Indeed there is no need to guarantee that the strain energy of the plate is coercive. Thus the whole space $H^2(\Omega)$ is the kinematically admissible set of the plate. If one takes now into account the relation (21), then $z \in K \subset H^2(\Omega)$, where K is a closed convex bounded subset of $H^2(\Omega)$ and the problem has the following form:

Find $\zeta \in K$ such as to satisfy

$$a(\zeta, z-\zeta) + \int_{\Omega'} j^0(\zeta, z-\zeta)d\Omega \geq (f, z-\zeta) \qquad \forall z \in K. \tag{22}$$

Here $a(\cdot,\cdot)$ is the elastic energy of the Kirchoff plate, i.e.

$$a(\zeta, z) = k \int_{\Omega} [(1-\nu)\, \zeta_{,\alpha\beta}\, z_{,\alpha\beta} + \nu\, \Delta\zeta\, \Delta z] d\Omega \qquad \alpha, \beta = 1, 2, \tag{23}$$

where $k = Eh^3/12(1-\nu^2)$ is the bending rigidity of the plate with E and ν the modulus of elasticity and the Poisson ratio respectively, and h is its thickness. Moreover j is the binding energy of the adhesive which is a locally Lipschitz function on $H^2(\Omega)$ and $f \in L^2(\Omega)$ denotes the external forces. Furthermore, if j fulfills the growth condition (j) then, taking into consideration that $a(\cdot,\cdot)$ appearing in (23) is continuous monotone, we can deduce, by applying Theorem 2, the existence of a solution of the problem (22).

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Double Eigenvalue Hemivariational Inequalities with Non-Locally Lipschitz Energy Functional

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Abstract

We give an existence result for a double eigenvalue problem in Hemivariational Inequalities whose energetic functional is not locally Lipschitz. It is used a finite dimensional approach based on Kakutani's fixed point theorem.

Key Words: eigenvalue problem, generalized gradient, hemivariational inequality, linear elasticity

1 Introduction and formulation of the problem

The concept of hemivariational inequality has been introduced by P.D. Panagiotopoulos as a natural extension of the variational inequalities to the case of nonconvex functionals. This extension is strongly motivated by many problems arising in Mechanics, Engineering or Economics. For a comprehensive overview on this subject we refer to the monographs [P] and [NP].

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In this paper we deal with a new type of hemivariational inequalities called "double eigenvalue problems" which has been introduced by D. Motreanu and P.D. Panagiotopoulos in a paper where there are considered three different approaches: minimization, minimax methods and (sub)critical theory on the sphere (see [MP]). Other results on this type of hemivariational inequalities can be found in [BMP] (multiplicity results) and [BPR] (a perturbation result).

Let V be a Hilbert space and let $\Omega \subset \mathbf{R}^m$ be an open bounded subset of $\mathbf{R}^m, m \geq 1$, with $\partial\Omega$ sufficiently smooth. We shall suppose that V is compactly embedded into $L^p(\Omega; \mathbf{R}^N), N \geq 1$, for some $p \in (1, +\infty)$. In particular, the continuity of this embedding implies the existence of a constant $C_p(\Omega) > 0$ such that

$$(*) \qquad \|u\|_{L^p} \leq C_p(\Omega) \cdot \|u\|_V, \text{ for all } \quad u \in V,$$

where by $\|\cdot\|_{L^p}$ and $\|\cdot\|_V$ we have denoted the norms in $L^p(\Omega; \mathbf{R}^N)$ and V respectively. Throughout the paper the symbols V^*, $(\cdot,\cdot)_V$, and $\langle\cdot,\cdot\rangle$ will denote the dual space of V, the inner product on V and the duality pairing over $V^* \times V$, respectively. We suppose that $V \cap L^\infty(\Omega; \mathbf{R}^N)$ is dense in V. Let $a_1, a_2 : V \times V \to \mathbf{R}$ be two bilinear and continuous forms on V which are coercive in the sense that there exist two real valued functions $c_1, c_2 : \mathbf{R}_+ \to \mathbf{R}_+$, with $\lim\limits_{r\to\infty} c_i(r) = +\infty$, such that for all $v \in V$

$$a_i(v,v) \geq c_i(\|v\|_V) \cdot \|v\|_V, \;\; i = 1,2.$$

We denote by $A_1, A_2 : V \to V$ the operators associated to the forms considered above, defined by

$$\langle A_i u, v\rangle = a_i(u,v), \;\; i = 1,2.$$

The operators A_1 and A_2 are linear, continuous and coercive in the sense that for each $i = 1,2$ we have

$$(A_i u, u)_V \geq c_i(\|u\|_V) \cdot \|u\|_V, \text{ for all } \; u \in V.$$

In addition we shall suppose that the operators A_1 and A_2 are weakly continuous, i.e., if $u_n \rightharpoonup u$, weakly in V then $A_i u_n \rightharpoonup A_i u$, also weakly in V, for each $i = 1,2$. Let us now consider two bounded selfadjoint linear and weakly continuous operators $B_1, B_2 : V \to V$. Let $j : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory function which is locally Lipschitz in the second variable for a.e. $x \in \Omega$. Thus, we can define the directional derivative

$$j^0(x;\xi,\eta) = \limsup_{[h,\lambda]\to[0,0^+]} \frac{j(x,\xi+h+\lambda\eta) - j(x,\xi+h)}{\lambda}, \text{ for } \; \xi,\eta \in \mathbf{R}^N,$$

and the generalized gradient of Clarke [C]

$$\partial j(x;\xi) = \{\eta \in \mathbf{R}^N \; : \; \eta \cdot \gamma \leq j^0(x,\xi,\gamma)\, , \forall \gamma \in \mathbf{R}^N\}\, ,$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbf{R}^N$. Here, the symbol " $\cdot$ " means the inner product on $\mathbf{R}^N$.

In order to ensure the integrability of $j(\cdot, u(\cdot))$ and $j^0(\cdot; u(\cdot), v(\cdot))$ for any $u, v \in V \cap L^\infty(\Omega; \mathbf{R}^N)$ we admit the existence of a function $\beta : \Omega \times \mathbf{R}_+ \to \mathbf{R}$ fulfilling the conditions

(β_1) $\beta(\cdot, r) \in L^1(\Omega)$, for each $r \geq 0$;
(β_2) if $r_1 \leq r_2$ then $\beta(x, r_1) \leq \beta(x, r_2)$, for almost all $x \in \Omega$, and such that

$$|j(x,\xi) - j(x,\eta)| \leq \beta(x,r) \cdot |\xi - \eta|, \forall\, \xi, \eta \in B(O,r), r \geq 0, \tag{1}$$

where $B(O,r) = \{\xi \in \mathbf{R}^N \ : \ |\xi| \leq r\}$, "$|\cdot|$" denoting the norm in $\mathbf{R}^N$.

Concerning the conditions above, it is important to point out that in the homogenous case (when j is not depending explicitely on $x \in \Omega$) they are negligible (see also [NP], p. 146).

Let $1 \leq s < p$ and let $k : \Omega \to \mathbf{R}_+$ and $\alpha : \Omega \times \mathbf{R}_+$ be two functions satisfying the assumptions:

$$k(\cdot) \in L^q(\Omega), \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \tag{2}$$

$$\alpha(\cdot, r) \in L^t(\Omega), \text{ for each } r > 0, \text{ where } t = \frac{p}{p-s} \tag{3}$$

and

$$\text{if } 0 < r_1 \leq r_2 \text{ then } \alpha(x, r_1) \leq \alpha(x, r_2), \text{ for almost all } x \in \Omega. \tag{4}$$

We shall impose the following directional growth conditions:

$$j^0(x, \xi, -\xi) \leq k(x) \cdot |\xi|, \text{ for all } \xi \in \mathbf{R}^N \text{ and a.e. } x \in \Omega; \tag{5}$$

$$j^0(x, \xi, \eta - \xi) \leq \alpha(x,r)\left(1 + |\xi|^s\right), \text{ for all } \xi, \eta \in \mathbf{R}^N, \text{ with } \eta \in B(O,r), r > 0, \text{ and a.e. } x \in \Omega. \tag{6}$$

REMARKS: 1. We must pay attention to the fact that the growth conditions (5) and (6) do not ensure the finite integrability of $j(\cdot, u(\cdot))$ and $j^0(\cdot; u(\cdot), v(\cdot))$ in Ω for any $u, v \in V$. We can remark, also, that they do not guarantee that the functional $J : V \to \mathbf{R}$ given by

$$J(v) = \int_\Omega j(x, v(x))dx,$$

is locally Lipschitz on V. In fact, (5) and (6) do not allow us to conclude even that the effective domain of J coincides with the whole space V.

2. Notice that we do not impose any coerciveness assumption on the operators B_i $(i = 1, 2)$, as done in [MP], Section 4, for the case of a double eigenvalue problem on a sphere. We suppose however that these operators satisfy the additional hypothesis of weak continuity.

Let us consider two nonlinear monotone and demicontinuous operators $C_1, C_2 : V \to V$. We are ready to consider the following double eigenvalue problem:

(P) Find $u_1, u_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$a_1(u_1, v_1) + a_2(u_2, v_2) + (C_1(u_1), v_1)_V + (C_2(u_2), v_2)_V +$$

$$+ \int_\Omega j^0(x; (u_1 - u_2)(x), (v_1 - v_2)(x))dx \geq \lambda_1 (B_1 u_1, v_1)_V + \lambda_2 (B_2 u_2, v_2)_V, \forall v_1, v_2 \in V.$$

From Remark 1 we derive that in order to find a solution for the problem (P) we cannot follow the classical technique of Clarke [C] and for this reason, the problem (P) is a nonstandard one. First of all we have to point out what we shall mean by solution of the problem considered above.

DEFINITION 1. *We say that an element* $(u_1, u_2, \lambda_1, \lambda_2) \in V \times V \times \mathbf{R} \times \mathbf{R}$ *is a solution of (P) if there exists* $\chi \in L^1(\Omega; \mathbf{R}^N) \cap V$ *such that*

$$a_1(u_1, v_1) + a_2(u_2, v_2) + (C_1(u_1), v_1)_V + (C_2(u_2), v_2)_V +$$

$$+ \int_\Omega \chi(x) \cdot (v_1 - v_2)(x)dx = \lambda_1 (B_1 u_1, v_1)_V + \lambda_2 (B_2 u_2, v_2)_V, \ \forall v_1, v_2 \in V \cap L^\infty(\Omega; \mathbf{R}^N) \tag{7}$$

and

$$\chi(x) \in \partial j(x; (u_1 - u_2(x)), \text{ for a.e. } x \in \Omega. \tag{8}$$

The aim of this paper is to prove the following existence result concerning the double eigenvalue problem (P).

THEOREM 1. *We assume that the hypotheses considered in this section are fulfilled. Then the double eigenvalue problem (P) has at least one solution.*

The difficulties mentioned in the Remark 1 will be surmounted by employing the Galerkin approximation method combined with the finite intersection property. For the treatment of finite dimensional problem we shall use Kakutani's fixed point theorem for multivalued mappings. This technique has been introduced by Naniewicz and Panagiotopoulos (see [NP]).

2 A finite dimensional approach

Let Λ be the family of all finite dimensional subspaces F of $V \cap L^\infty(\Omega; \mathbf{R}^N)$, ordered by inclusion. For any $F \in \Lambda$ we formulate the following finite dimensional problem

(P_F) Find $u_{1F}, u_{2F} \in F, \lambda_1, \lambda_2 \in \mathbf{R}$ *and* $\chi_F \in L^1(\Omega; \mathbf{R}^N)$ *such that*

$$a_1(u_{1F}, v_1) + a_2(u_{2F}, v_2) + (C_1(u_{1F}), v_1)_V + (C_2(u_{2F}), v_2)_V +$$

$$+ \int_\Omega \chi_F(x) \cdot (v_1 - v_2)(x)dx = \lambda_1 (B_1 u_{1F}, v_1)_V + \lambda_2 (B_2 u_{2F}, v_2)_V, \ \forall v_1, v_2 \in F \tag{9}$$

and

$$\chi_F(x) \in \partial j(x; (u_{1F} - u_{2F})(x)), \text{ for a.e. } x \in \Omega. \tag{10}$$

Let $\Gamma_F : F \to 2^{L^1(\Omega;\mathbf{R}^N)}$ defined by

$$\Gamma_F(v_F) = \{\Psi \in L^1(\Omega;\mathbf{R}^N) : \int_\Omega \Psi w dx \leq \int_\Omega j^0(x; v_F(x), w(x))dx, \ \forall w \in L^\infty(\Omega;\mathbf{R}^N)\}.$$

It is immediately that if $\Psi \in \Gamma_F(v_F)$ then we have $\Psi(x) \in \partial j(x; v_F(x))$, for a.e. $x \in \Omega$. Let $v_F \in F$ for some $F \in \Lambda$. It is proved in [N] (see Lemma 3.1) that $\Gamma(v_F)$ is a nonempty convex and weakly compact subset of $L^1(\Omega;\mathbf{R}^N)$. For $F \in \Lambda$, we shall denote by $i_F : F \to V$ and by $i_F^* : V^* \to F^*$ the inclusion and the dual projection mappings respectively. Throughout, by $\langle \cdot, \cdot \rangle_F$ we mean the duality pairing over $F^* \times F$. Let us define $\gamma_F : L^1(\Omega;\mathbf{R}^N) \to F^*$, by

$$\langle \gamma_F \Psi, v \rangle_F = \int \Psi \cdot v dx, \ \forall v \in F.$$

We consider the map $T_F : F \to 2^{F^*}$ given by

$$T_F(v_F) = \gamma_F \Gamma_F(v_F).$$

The main properties of T_F are pointed out by the following result which has been established in [N].

LEMMA 1. *For each $v_F \in F, T_F(v_F)$ is a nonempty bounded closed convex subset of F^*. Moreover, T_F is upper semicontinuous as a map from F into 2^{F^*}.*

We are now prepared to formulate the existence result for the finite dimensional problem (P_F).

THEOREM 2. *Suppose that the hypotheses made in Section 1 are fulfilled. Then, for each $F \in \Lambda$, there exist $u_{1F}, u_{2F} \in F, \lambda_1, \lambda_2 \in \mathbf{R}$ and $\chi_F \in L^1(\Omega;\mathbf{R}^N)$ which solve the problem (P_F). Moreover, there exists a positive constant M, independent by F such that*

$$\|u_{1F}\|_V + \|u_{2F}\|_V \leq M. \tag{11}$$

Proof. In what follows we shall be able to find a solution of the problem (P_F) by restraining the searching area for $\lambda_i, \ i \in \{1,2\}$ on the class of all those numbers $\lambda_1, \lambda_2 \in \mathbf{R}$ which satisfy the relation

$$\delta := \inf_{w_1, w_2 \in V \cap L^\infty(\Omega;\mathbf{R}^N)} \frac{\sum_{i=1}^{2} [(C_i(w_i), w_i)_V - \lambda_i \|B_i\| \|w_i\|_V^2]}{\|u_1\| + \|u_2\|} > -\infty. \tag{12}$$

Define $A_{1F} = i_F^* A_1 i_F$, $A_{2F} = i_F^* A_2 i_F$, and let $\overline{G} : V \times V \to V$ be the map given by

$$\overline{G}(v_1, v_2) = v_1 - v_2.$$

Fix $F \in \Lambda$. We denote by G the map $\overline{G}$ restricted to $F \times F$. Let us consider the multivalued mapping $\Delta : F \times F \to 2^{F^* \times F^*}$, defined by

$$\Delta(u_1, u_2) = (A_{1F}u_1 + (C_1(u_1), u_1)_V - \lambda_1 (B_1 u_1, \cdot),$$

$$A_{2F}u_2 + (C_2(u_2), \cdot)_V - \lambda_2 (B_2 u_2, \cdot)_V) + (G^* \circ T_F \circ G)(u_1, u_2),$$

where by $(G^* \circ T_F \circ G)(u_1, u_2)$ we mean the set

$$\{G^*(f) \ : \ f \in T_F(u_1 - u_2)\} \subset F^* \times F^*.$$

The first step is to prove the upper semicontinuity of $G^* \circ T_F \circ G$. For this aim, let us consider $u_n^1 \to u_1, u_n^2 \to u_2$, strongly in F and $\Psi_n \in G^*(T_F(u_n^1 - u_n^2))$ converging strongly to $\Psi \in F^* \times F^*$. It must be proved that $\Psi \in G^*(T_F(u^1 - u^2))$. First we observe that G fulfills the set of conditions which permits to apply the theorem II.19 from [B]. From there we draw the conclusion that $\Re(G^*) = \{G^*\theta \ : \theta \in F^*\}$ is closed. This implies that $\Psi \in \Re(G^*)$ (we have used the fact that $\Psi_n \in \Re(G^*), \forall n \geq 1$ and $\Psi_n \to \Psi$ in $F^* \times F^*$). Thus we obtain the existence of a $\xi^* \in F^*$ such that $\Psi_n = G^*(\gamma_F \chi_n)$. We have

$$\langle G^*(\gamma_F \chi_n, (v, w)\rangle_{F \times F} \to \langle \Psi, (v, w)\rangle_{F \times F}, \text{ for all } v, w \in F,$$

which implies that $\langle \gamma_F \chi_n, v - w\rangle_F$ tends to $\langle \xi^*, v - w\rangle_F, \forall v, w \in F$ and thus, due to the fact that $\dim F < +\infty$ we get the strong convergence of $\gamma_F \chi_n$ to ξ^* in F^*. Since T_F is upper semicontinuous (see Lemma 1), we obtain that there exists $\chi \in \Gamma_F(u_1 - u_2)$ such that $\xi^* = \gamma_F \chi$. Thus, $\Psi = G^*(\gamma_F \chi)$, which means that $\Psi \in (G^* \circ T_F)(u_1 - u_2)$. This ends the proof of the upper semicontinuity of $G^* \circ T_F \circ G$.

On the other side, the weak continuity of A_1 and A_2 implies the continuity of A_{1F} and A_{2F} from F into F^*. The hypotheses on B_i and $C_i (i = 1, 2)$ and the above considerations lead us to the upper semicontinuty of Δ from $F \times F$ to $2^{F^* \times F^*}$. By using again Lemma 1 and the hypotheses made on B_i, C_i and A_i, we can simply derive that for each $(u_1, u_2) \in F \times F, \Delta(u_1, u_2)$ is a nonempty, bounded, closed and convex subset of $F^* \times F^*$. Moreover, from the coercivity of a_1 and a_2 and from the definition of T_F we have

$$\langle \Delta(u_1, u_2), (u_1, u_2)\rangle_{F \times F} \geq c_1(\|u_1\|_V)\|u_1\|_V + c_2(\|u_2\|_V)\|u_2\|_V + (C_1(u_1), u_1)_V + (C_2(u_2), u_2)_V -$$

$$-\lambda_1 \|B_1\| \cdot \|u_1\|_V^2 - \lambda_2 \|B_2\| \cdot \|u_2\|_V^2 + \int_\Omega \Psi(u_1 - u_2) dx,$$

where $\Psi \in \Gamma_F(u_1 - u_2)$. By (*) and (5) we obtain

$$\langle \Delta(u_1, u_2), (u_1, u_2)\rangle_{F \times F} \geq c_1(\|u_1\|_V)\|u_1\|_V + c_2(\|u_2\|_V)\|u_2\|_V + (C_1(u_1), u_1)_V + (C_2(u_2), u_2)_V -$$

$$-\lambda_1\|B_1\|\cdot\|u_1\|_V^2-\lambda_2\|B_2\|\cdot\|u_2\|_V^2-\int\limits_\Omega j^0(x;(u_1-u_2)(x),-(u_1-u_2)(x)dx\geq$$

$$\geq c_1(\|u_1\|_V)\|u_1\|_V+c_2(\|u_2\|_V)\|u_2\|_V+(C_1(u_1),u_1)_V+(C_2(u_2),u_2)_V-\lambda_1\|B_1\|\cdot\|u_1\|_V^2-$$
$$\lambda_2\|B_2\|\cdot\|u_2\|_V^2-C_p(\Omega)\|k\|_{L^q}\left(\|u_1\|_V+\|u_2\|_V\right).$$

Taking into account the relation (12) we easily obtain the coercivity of Δ. Thus, Δ fulfilles the conditions which allows us to apply Kakutani's fixed point theorem (see [BH], Proposition 10, p. 270). Thus $\Re(\Delta)=F^*\times F^*$, which implies the existence of $u_{1F},u_{2F}\in F$ such that $0\in\Delta(u_{1F},u_{2F})$. From the definition of Δ we have that there exists $\chi_F\in L^1(\Omega;\mathbf{R}^N)$ such that (9) and (10) hold. In order to prove the final part of Theorem 2 we use the estimates:

$$\lambda_1\|B_1\|u_{1F}\|_V^2+\lambda_2\|B_2\|u_{2F}\|_V^2\geq\lambda_1\left(B_1u_{1F},u_{1F}\right)_V+\lambda_2\left(B_2u_{2F},u_{2F}\right)_V=a_1(u_{1F},u_{1F})+$$

$$+a_2(u_{2F},u_{2F})+(C_1(u_{1F}),u_{1F})_V+(C_2(u_{2F}),u_{2F})_V+\int\limits_\Omega\chi_F(u_{1F}-u_{2F})dx\geq$$

$$\geq c_1(\|u_{1F}\|_V)\|u_{1F}\|_V+c_2(\|u_{2F}\|_V)\|u_{2F}\|_V+(C_1(u_{1F}),u_{1F})_V+(C_2(u_{2F}),u_{2F})_V-$$
$$-\int\limits_\Omega j^0(x;(u_{1F}-u_{2F})(x),-(u_{1F}-u_{2F})(x))dx.$$

Taking into account the relations (5) and (12) we get

$$\frac{c_1(\|u_{1F}\|_V)\|u_{1F}\|_V+c_2(\|u_{2F}\|_V)\|u_{2F}\|_V}{\|u_{1F}\|_V+\|u_{2F}\|_V}\leq C_p(\Omega)\|k\|_{L^q}-\delta,$$

which by the properties of c_1 and c_2 implies the existence of a positive constant M such that (11) holds.

LEMMA 2. *For every $F\in\Lambda$, let $u_{1F},u_{2F}\in F,\lambda_1,\lambda_2\in\mathbf{R}$ and $\chi_F)\in L^1(\Omega;\mathbf{R}^N)$ which solve the peoblem (P_F). Then the set $\{\chi_F\ :\ F\in\Lambda\}$ is weakly precompact in $L^1(\Omega;\mathbf{R}^N)$.*

Proof. The proof is based on the well-known Dunford-Petis theorem. We have to prove that for each $\epsilon>0$, a $\delta_\epsilon>0$ may be determined such that, for any $\omega\subset\Omega$ with $meas(\omega)<\delta_\epsilon$,

$$\int\limits_\omega|\chi_F|dx<\epsilon,\ F\in\Lambda.$$

Fix $r>0$ and let $\eta\in\mathbf{R}^N$ be such that $|\eta|\leq r$. From $\chi_F\in\partial j(x;(u_{1F}-u_{2F})(x))$, for a.e. $x\in\Omega$ we derive that

$$\chi_F\cdot(\eta-(u_{1F}-u_{2F})(x))\leq j^0(x;(u_{1F}-u_{2F})(x),\eta-(u_{1F}-u_{2F})(x)).$$

Taking into account the relation (6) it follows that

$$\chi_F(x)\cdot\eta\leq\chi_F(x)\cdot(u_{1F}-u_{2F})(x)+\alpha(x,r)\left(1+|u_{1F}(x)-u_{2F}(x)|^s\right),\ \text{for a.e. } x\in\Omega. \quad (13)$$

Let us denote by $\chi_{Fi}(x), i = 1, 2, \cdots, N$ the components of $\chi_F(x)$ and set

$$\eta(x) = \frac{r}{\sqrt{N}} \left(sgn\chi_{F1}(x), \cdots, sgn\chi_{Fn}(x) \right).$$

We can easily verify that $|\eta(x)| \le r$ a.e. $x \in \Omega$ and that

$$\chi_F(x) \cdot \eta(x) \ge \frac{r}{\sqrt{N}} \cdot |\chi_F(x)|.$$

From (13) we obtain

$$\frac{r}{\sqrt{N}} \cdot |\chi_F(x)| \le \chi_F(x) \cdot (u_{1F} - u_{2F})(x) + \alpha(x, r) \left(1 + |u_{1F}(x) - u_{2F}(x)|^s\right)$$

Integrating over $\omega \subset \Omega$ the above inequality yields

$$\int_\omega |\chi_F(x)| dx \le \frac{\sqrt{N}}{r} \int_\omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\omega)} \cdot meas(\omega)^{\frac{s}{p}} +$$

$$+ \frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\omega)} \cdot \|u_{1F} - u_{2F}\|^S_{L^p(\omega)}.$$

Thus, from (*) and (11) we obtain

$$\begin{aligned} \int_\omega |\chi_F(x)| dx &\le \frac{\sqrt{N}}{r} \int_\omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\Omega)} \cdot meas(\omega)^{\frac{s}{p}} + \\ &+ \frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\omega)} \cdot (C_p(\Omega))^S \cdot \|u_{1F} - u_{2F}\|^S_V \le \frac{\sqrt{N}}{r} \int_\omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \\ &+ \frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\Omega)} \cdot meas(\omega)^{\frac{s}{p}} + \frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\omega)} \cdot (C_p(\Omega))^S \cdot M^S. \end{aligned} \tag{14}$$

We shall continue by observing that (5) implies

$$\chi_F(x) \cdot (u_{1F}(x) - u_{2F}(x)) + k(x) \cdot (1 + |u_{1F}(x) - u_{2F}(x)|) \ge 0, \text{ for a.e. } x \in \Omega.$$

Thus we have

$$\int_\omega \left(\chi_F(x) \cdot (u_{1F} - u_{2F})(x) + k(x)(1 + |u_{1F}(x) - u_{2F}(x)|)\right) dx \le$$

$$\le \int_\Omega \left(\chi_F(x) \cdot (u_{1F} - u_{2F})(x) + k(x)(1 + |u_{1F}(x) - u_{2F}(x)|)\right) dx$$

and we derive that

$$\int_\omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx \le \int_\Omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x) dx + \|k\|_{L^q(\Omega)} \cdot C_p(\Omega) \cdot \|u_{1F} - u_{2F}\|_V +$$

$$+\|k\|_{L^q(\Omega)} \cdot meas(\Omega)^{\frac{1}{p}} \leq \int_\Omega \chi_F(x) \cdot (u_{1F} - u_{2F})(x)dx + \|k\|_{L^q(\Omega)} \cdot meas(\Omega)^{\frac{1}{p}} + \|k\|_{L^q(\Omega)} \cdot C_p(\Omega) \cdot M.$$

We have

$$\int_\Omega \chi_F(u_{1F} - u_{2F})dx = -\left(A_1 u_{1F}, u_{1F}\right)_V - \left(A_2 u_{2F}, u_{2F}\right)_V -$$

$$-\left(C_1(u_{1F}), u_{1F}\right)_V - \left(C_2(u_{2F}), u_{2F}\right)_V + \lambda_1 \left(B_1 u_{1F}, u_{1F}\right)_V + \lambda_2 \left(B_2 u_{2F}, u_{2F}\right)_V.$$

Taking into account that C_i are monotone operators and that A_i, being weakly continuous maps bounded sets into bounded sets, the relation

$$\int_\Omega \chi_F(u_{1F} - u_{2F})dx \leq \sum_{i=1}^{2} \{\|A_i\| \|u_{iF}\|_V^2 + \lambda_i \|B_i\| \|u_{iF}\|_V^2 - (C_i(u_{iF}), u_{iF})_V\},$$

imply that there exists a positive constant $\tilde{C}$ such that

$$\int_\Omega \chi_F(u_{1F} - u_{2F})dx \leq \tilde{C}. \tag{15}$$

Now, from (14) and (15) we obtain

$$\int_\omega |\chi_F(x)|dx \leq \frac{\sqrt{N}}{r} \cdot C + \frac{\sqrt{N}}{r} \cdot \|\alpha(\cdot, r)\|_{L^{q'}(\Omega)} \cdot meas(\omega)^{\frac{s}{p}} + \tag{16}$$

$$+ \frac{\sqrt{N}}{r} \cdot \|\alpha(\cdot, r)\|_{L^{q'}(\omega)} \cdot (C_p(\Omega))^S \cdot M^S,$$

where we have denoted

$$C := \tilde{C} + \|k\|_{L^q(\Omega)} \cdot meas(\Omega)^{\frac{1}{p}} + \|k\|_{L^q(\Omega)} \cdot C_p(\Omega) \cdot M.$$

Let $\epsilon > 0$. We choose $r > 0$ such that $\frac{\sqrt{N}}{r} \cdot C < \frac{\epsilon}{2}$. Since $\alpha(\cdot, r) \in L^{q'}(\Omega)$ we can determine $\delta_\epsilon > 0$ small enough such that if $meas(\omega) < \delta_\epsilon$, we have

$$\frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\Omega)} \cdot meas(\omega)^{\frac{s}{p}} + \frac{\sqrt{N}}{r} \|\alpha(\cdot, r)\|_{L^{q'}(\omega)} \cdot (C_p(\Omega))^S \cdot M^S < \frac{\epsilon}{2}.$$

By the relation (16) it follows that

$$\int_\omega |\chi_F(x)|dx \leq \epsilon,$$

for any $\omega \subset \Omega$ with $meas(\omega) < \delta_\epsilon$. This means that the weak precompactness of $\{\chi_F \ : \ F \in \Lambda\}$ in $L^1(\Omega; \mathbf{R}^N)$ is established.

3 Proof of Theorem 1

We are ready to prove Theorem 1, which is our main existence result. We shall follow a procedure introduced by Z. Naniewicz and P.D. Panagiotopoulos (see, for example [NP]). For every $F \in \Lambda$ let

$$W_F = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} \{(u_{1F'}, u_{2F'}, \chi_{F'})\} \subset V \times V \times L^1(\Omega; \mathbf{R}^N),$$

with $(u_{1F'}, u_{2F'}, \chi_{F'})$ being a solution of $(P_{F'})$. Moreover, let

$$Z = \bigcup_{F \in \Lambda} \{\chi_F\} \subset L^1(\Omega; \mathbf{R}^N).$$

Denoting by $weakcl(W_F)$ the weak closure of W_F in $V \times V \times L^1(\Omega; \mathbf{R}^N)$ and by $weakcl(Z)$ the weak closure of Z in $L^1(\Omega; \mathbf{R}^N)$ we obtain, taking into account the relation (12)

$$weakcl(W_F) \subset B_V(O, M) \times B_V(O, M) \times weakcl(Z), \text{ for every } F \in \Lambda.$$

Since V is reflexive it follows that $B_V(O, M)$ is weakly compact in V. Using Lemma 2 we get that the family $\{weakcl(W_F) \ : \ F \in \Lambda\}$ is contained in a wekly compact set of $V \times V \times L^1(\Omega; \mathbf{R}^N)$. It follows that this family has the finite intersection property and we may infer that

$$\bigcap_{F \in \Lambda} weakcl(W_F) \neq \emptyset$$

We choose (u_1, u_2, χ) belonging to the nonempty set above. In what follows we shall prove that this is the searched solution for the problem (P).

Let $v_1, v_2 \in L^\infty(\Omega; \mathbf{R}^N)$ and let F be an element of Λ such that $(v_1, v_2) \in F \times F$. We note that such an F exists, for example we can take $F = span\{v_1, v_2\}$. Since $(u_1, u_2, \chi) \in \bigcap_{F \in \Lambda} weakcl(W_F)$ it follows that there exists a sequence $\{(u_{1F_n}, u_{2F_n}, \chi_{F_n})\}$ in W_F, simply denoted by (u_{1n}, u_{2n}, χ_n) converging weakly to (u_1, u_2, χ) in $V \times V \times L^1(\Omega; \mathbf{R}^N)$. We have $u_{in} \rightharpoonup u_i$, weakly in $V (i = 1, 2)$ and $\chi_n \rightharpoonup \chi$, weakly in $L^1(\Omega; \mathbf{R}^N)$. Since (u_{1n}, u_{2n}, χ_n) is a solution of (P_F) we get

$$\langle A_1 u_{1n}, v_1 \rangle_V + \langle A_2 u_{2n}, v_2 \rangle_V + (C_1(u_{1n}), v_1)_V + (C_2(u_{2n}), v_2)_V +$$
$$+ \int_\Omega \chi_n (v_1 - v_2) dx = \lambda_1 (B_1 u_{1n}, v_1)_V + \lambda_2 (B_2 u_{2n}, v_2)_V$$

The hypotheses on $A_i, B_i, C_i (i = 1, 2)$ and the convergenses above imply the equality

$$\sum_{i=1}^{2} \{\langle A_i u_i, v_i \rangle_V + (C_i(u_i), v_i)_V - \lambda_i (B_i u_i, v_i)_V\} + \int_\Omega \chi (v_1 - v_2) dx = 0,$$

which is satisfied for any $v_1, v_2 \in V \cap L^\infty(\Omega; \mathbf{R}^N)$. By the density of $V \cap L^\infty(\Omega; \mathbf{R}^N)$ in V we draw the conclusion that the relation (7) is valid for any $v_1, v_2 \in V$.

In what follows we shall prove the relation (8). Due to the compact embedding $V \subset L^p(\Omega;\mathbf{R}^N)$ it results from the weak convergences $u_{in} \rightharpoonup u_i$ in V that we have

$$u_{in} \to u_i \text{ strongly in } L^p(\Omega;\mathbf{R}^N), \text{ for each } i = 1,2\,.$$

So, by passing eventually to a subsequence we have

$$u_{in} \to u_i \text{ a.e. in } \Omega.$$

From the Egoroff theorem we obtain that for any $\epsilon > 0$ a subset $\omega \subset \Omega$ with $meas(\omega) < \epsilon$ can be determined such that for each $i \in \{1,2\}$

$$u_{in} \to u_i \text{ uniformly on } \Omega\backslash\omega,$$

with $u_i \in L^\infty(\Omega\backslash\omega;\mathbf{R}^N)$ for every $i \in \{1,2\}$. Let $v \in L^\infty(\Omega\backslash\omega;\mathbf{R}^N)$ be arbitrarily chosen. The Fatou's lemma now implies that for any $\mu > 0$ there exists $\delta_\mu > 0$ and a positive integer N_μ such that

$$\int_{\Omega\backslash\omega} \frac{j(x;(u_{1n}-u_{2n})(x)-\theta+\lambda v(x)) - j(x;(u_{1n}-u_{2n})(x)-\theta)}{\lambda}dx \leq \tag{17}$$

$$\leq \int_{\Omega\backslash\omega} j^0(x;(u_1-u_2)(x),v(x))dx + \mu,$$

for every $n \geq N_\mu$, $|\theta| < \delta_\mu$ and $\lambda \in (0,\delta_\mu)$. Taking into account that $\chi_n \in \partial j(x;(u_{1n}-u_{2n})(x))$ for a.e. $x \in \Omega$ we have

$$\int_{\Omega\backslash\omega} \chi_n(x)\cdot v(x)dx \leq \int_{\Omega\backslash\omega} j^0(x;(u_{1n}-u_{2n})(x),v(x))dx. \tag{18}$$

Passing to the limit as $\lambda \to 0$ in (17) and employing the relation (18) it follows that

$$\int_{\Omega\backslash\omega} \chi_n(x)\cdot v(x)dx \leq \int_{\Omega\backslash\omega} j^0(x;(u_1-u_2)(x),v(x))dx + \mu.$$

From the relation above and the weak convergence of χ_n to χ in $L^1(\Omega;\mathbf{R}^N)$ we derive that

$$\int_{\Omega\backslash\omega} \chi(x)\cdot v(x)dx \leq \int_{\Omega\backslash\omega} j^0(x;(u_1-u_2)(x),v(x))dx + \mu.$$

Since $\mu > 0$ was chosen arbitrarily ,

$$\int_{\Omega\backslash\omega} \chi(x)\cdot v(x)dx \leq \int_{\Omega\backslash\omega} j^0(x;(u_1-u_2)(x),v(x))dx,\ \forall v \in L^\infty(\Omega\backslash\omega;\mathbf{R}^N).$$

The last inequality implies that

$$\chi(x) \in \partial j(x;(u_1-u_2)(x)), \text{ for a.e. } x \in \Omega\backslash\omega,$$

where $meas(\omega) < \epsilon$. Since $\epsilon > 0$ was chosen arbitrarily we have that

$$\chi(x) \in \partial j(x;(u_1-u_2)(x)), \text{ for a.e. } x \in \Omega,$$

which means that the relation (8) holds. The proof of Theorem 1 is now complete.

4 Application: The Multiple Loading Buckling

We consider two elastic beams (linear elasticity) of length l measured along the axis Ox of the coordinate system yOx, and with the same cross-section. The beams, numbered here by $i = 1, 2$, are simply supported at their ends $x = 0$ and $x = l$. On the interval (l_1, l_2), $l_1 < l_2 < l$, they are connected with an adhesive material of negligible thickness. The displacements of the i-th beam are denoted by $x \to u_i(x)$, $i = 1, 2$, and the behaviour of the adhesive material is described by a nonmonotone possibly multivalued law between $-f(x)$ and $[u(x)]$, where $x \to f(x)$ denotes the reaction force per unit length vertical to the Ox axis, due to the adhesive material (cf. [P] p.87 and [NP] p.110) and $[u] = u_1 - u_2$ is the relative deflection of the two beams. Recall that u_i is referred to the middle line of the beam i (the dotted lines in Fig. 1) and that each beam has constant thickness which remains the same after the deformation. The adhesive material can sustain a small tensile force before rupture (debonding). In Fig. 1 a rupture of zig-zag brittle type is depicted in the $(-f, u)$ diagramm. The beams are assumed to have the same moduli of elasticity E and let I be the moment of inertia of them. The sandwich beam is subjected to the compressive forces P_1 and P_2 and we want to determine the buckling loading of it. This problem is yet open problem in Engineering. From the large deflection theory of beams we may write the following relations which describe the behaviour of the i-th beam:

$$u_i''''(x) + \frac{1}{a_i^2} u_i''(x) = f_i(x) \qquad \text{on } (0, l)\,; \tag{19}$$

$$u_i(0) = u_i(l) = 0\,, \qquad u_i''(0) = u_i''(l) = 0 \qquad i = 1, 2\,. \tag{20}$$

Here $a_i^2 := IE/P_i$. We assume that the $(-f, [u])$ graph results from a non locally Lipschitz function $j : \mathbf{R} \to \overline{\mathbf{R}}$ such that

$$-f(x) \in \partial j([u(x)])\,, \quad \forall x \in (l_1, l_2)\,, \tag{21}$$

where ∂ denotes the generalized gradient of Clarke. We set

$$V := H^2(\Omega) \cap H_0^1(\Omega) \qquad \Omega = (0, l)\,. \tag{22}$$

It is a Hilbert space with the inner product (see [DL], p. 216, Lemma 4.2) $a(u, v) := \int_0^l u''(x)v''(x)dx$.

Let $L : V \to V^\star$ be the linear operator defined by

$$\langle Lu, v\rangle := \int_0^l u'(x)v'(x)dx\,, \quad \forall u, v \in V\,. \tag{23}$$

We observe easily that L is bounded, weak continuous and satisfies

$$\langle Lu, v\rangle = \langle Lv, u\rangle\,, \qquad \text{for all } u, v \in V\,.$$

The superpotential law (21) implies that

$$j^0([u(x)];y) \geq -f(x)y\,, \quad \forall x \in (l_1,l_2)\,, \forall y \in \mathbf{R}\,. \tag{24}$$

Multiplying (19) by $v_i(x)-u_i(x)$, integrating over $(0,l)$ and adding the resulting relations for $i = 1,2$, implies by taking into account the boundary condition (20), the hemivariational inequality

$$u=\{u_1,u_2\}\in V\times V\,,\quad \sum_{i=1}^{2}\int_0^l u_i''(x)[v_i''(x)-u_i''(x)]dx-\sum_{i=1}^{2}\frac{1}{a_i^2}\int_0^l u_i'(x)[v_i'(x)-u_i'(x)]dx$$

$$+\int_{l_1}^{l_2} j^0([u(x)];[v(x)]-[u(x)])dx \geq 0\,, \qquad \forall v=\{v_1,v_2\}\in V\times V\,. \tag{25}$$

Thus buckling of the beam occurs if $\lambda_i := 1/a_i^2$ $(i = 1,2)$ is an eigenvalue for the following hemivariational inequality

$$\sum_{i=1}^{2} a_i(u_i,v_i-u_i)-\sum_{i=1}^{2}\lambda_i\,\langle u_i,v_i-u_i\rangle+\int_{l_1}^{l_2} j^0([u(x)];[v(x)]-[u(x)])dx \geq 0\,, \tag{26}$$

for all $v=\{v_1,v_2\}\in V\times V$. According to the Theorem 1 the present problem admits at least one solution $\{u_1,u_2,\lambda_1,\lambda_2\}$, provided that j fulfills the growth assumption given in Sect. 1, *i.e.*, (1), (5) and (6).

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EXISTENCE RESULTS FOR INEQUALITY PROBLEMS WITH LACK OF CONVEXITY*

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Abstract. We establish several existence results of Hartman-Stampacchia type for hemivariational inequalities on bounded and convex sets in a real reflexive Banach space. We also study the cases of coercive and noncoercive variational-hemivariational inequalities.

1 Introduction

The study of variational inequality problems began around 1965 with the pioneering works of G. Fichera, J.-L. Lions and G. Stampacchia (see [4], [7]). The connection of the theory of variational inequalities with the notion of subdifferentiability of convex analysis was achieved by J.J. Moreau (see [8]) who introduced the notion of convex superpotential which permitted the formulation and the solving of a wide ranging class of complicated problems in mechanics and engineering which could not until then be treated correctly by the methods of classical bilateral mechanics. All the inequality problems treated to the middle of the ninth decade were related to convex energy functions and therefore were firmly bound with monotonicity; for instance, only monotone, possibly multivalued boundary conditions and stress-strain laws could be studied. In order to overcome this limitation, P.D. Panagiotopoulos introduced in [14], [15] the notion of nonconvex superpotential by using the generalized gradient of F.H. Clarke. Due to the lack of convexity new types of variational expressions were obtained. These are the so-called *hemivariational inequalities* and they are no longer connected with monotonicity. Generally speaking, mechanical problems involving nonmonotone, possibly multivalued stress-strain laws or boundary conditions derived by nonconvex superpotentials lead to hemivariational inequali-

*This paper is dedicated to the memory of Professor P.D. Panagiotopoulos ties. Moreover, while in the convex case the static variational inequalities generally give rise to minimization problems for the potential or the complementary energy, in the nonconvex case the problem of substationarity of the potential or the complementary energy at an equilibrium position emerges.

Throughout this paper X will denote a real reflexive Banach space, (T, μ) will be a measure space of positive and finite measure and $A : X \to X^\star$ will stand for a nonlinear operator. We also assume that there are given $m \in \mathbf{N}$, $p \geq 1$ and a compact mapping $\gamma : X \to L^p(T, \mathbf{R}^m)$. We shall denote by p' the conjugated exponent of p. If $\varphi : X \to \mathbf{R}$ is a locally Lipschitz functional then $\varphi^0(u; v)$ will stand for the Clarke derivative of φ at $u \in X$ with respect to the direction $v \in X$, that is

$$\varphi^0(u; v) = \limsup_{\substack{w \to u \\ \lambda \downarrow 0}} \frac{\varphi(w + \lambda v) - \varphi(w)}{\lambda}.$$

Accordingly, Clarke's generalized gradient $\partial\varphi(u)$ of φ at u is defined by

$$\partial\varphi(u) = \{\xi \in X^\star ;\ \langle \xi, v \rangle \leq \varphi^0(u; v),\ \forall v \in X\}.$$

Let $j : T \times \mathbf{R}^m \to \mathbf{R}$ be a function such that the mapping

$$j(\cdot, y) : T \to \mathbf{R} \qquad \text{is measurable, for every } y \in \mathbf{R}^m. \tag{1}$$

We assume that at least one of the following conditions hold: either there exists $k \in L^{p'}(T, \mathbf{R})$ such that

$$|j(x, y_1) - j(x, y_2)| \leq k(x)\, |y_1 - y_2|, \qquad \forall x \in T, \forall y_1, y_2 \in \mathbf{R}^m, \tag{2}$$

or

$$\text{the mapping } j(x, \cdot) \text{ is locally Lipschitz, } \forall x \in T, \tag{3}$$

and there exists $C > 0$ such that

$$|z| \leq C(1 + |y|^{p-1}), \qquad \forall x \in T, \forall y_1, y_2 \in \mathbf{R}^m, \forall z \in \partial_y j(x, y). \tag{4}$$

Let K be a nonempty closed, convex subset of X, $f \in X^\star$ and $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a convex, lower semicontinuous functional such that

$$D(\Phi) \cap K \neq \emptyset. \tag{5}$$

Throughout this paper $\langle \cdot, \cdot \rangle$ will denote the duality pairing between $X^\star$ and X.

2 The generalized Hartman-Stampacchia theorem for variational-hemivariational inequality problems

Consider the following inequality problem:

Find $u \in K$ such that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + \int_T j^0(x, \gamma(u(x)); \gamma(v(x) - u(x)))d\mu \geq 0\,, \qquad \forall v \in K\,, \quad (6)$$

where γ denotes the prescribed canonical mapping from X into $L^p(T, \mathbf{R}^m)$.

The following two situations are of particular interest in applications:
(i) $T = \Omega$, $\mu = dx$, $X = W^{1,q}(\Omega, \mathbf{R}^m)$ and $\gamma : X \to L^p(\Omega, \mathbf{R}^m)$, with $p < q^*$, is the Sobolev embedding operator;
(ii) $T = \partial\Omega$, $\mu = d\sigma$, $X = W^{1,p}(\Omega, \mathbf{R}^m)$ and $\gamma = i \circ \eta$, where $\eta : X \to W^{1-\frac{1}{p},p}(\partial\Omega, \mathbf{R}^m)$ is the trace operator and $i : W^{1-\frac{1}{p},p}(\partial\Omega, \mathbf{R}^m) \to L^p(\partial\Omega, \mathbf{R}^m)$ is the embedding operator.

A direct application of the Knaster-Kuratowski-Mazurkiewicz (KKM, in short) principle (see [6] or [3]) leads to the following basic auxiliary result:

Lemma 1 *Let K be a nonempty, bounded, closed, convex subset of X, $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a convex, lower semicontinuous functional such that (5) holds. Consider a Banach space Y such that there exists a linear and compact mapping $L : X \to Y$ and let $J : Y \to \mathbf{R}$ be an arbitrary locally Lipschitz function. Suppose in addition that the mapping $K \ni v \longmapsto \langle Av, v - u\rangle$ is weakly lower semicontinuous, for every $u \in K$.*

Then, for every $f \in X^\star$, there exists $u \in K$ such that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0\left(L(u), L(v - u)\right) \geq 0\,, \forall v \in K\,. \quad (7)$$

Proof. Let us define the set-valued mapping $G : K \cap D(\Phi) \to 2^X$ by

$$G(x) = \{v \in K \cap D(\Phi); \langle Av - f, v - x\rangle - J^0(L(v); L(x) - L(v)) + \Phi(v) - \Phi(x) \leq 0\}\,.$$

We claim that the set $G(x)$ is weakly closed. Indeed, if $G(x) \ni v_n \rightharpoonup v$ then, by our hypotheses,

$$\langle Av, v - x\rangle \leq \liminf_{n\to\infty}\langle Av_n, v_n - x\rangle$$

and

$$\Phi(v) \leq \liminf_{n\to\infty} \Phi(v_n)\,.$$

Moreover, $L(v_n) \to L(v)$ and thus, by the upper semi-continuity of J^0 (see [2]), we also obtain

$$\limsup_{n\to\infty} J^0\left(L(v_n); L(x - v_n)\right) \leq J^0\left(L(v); L(x - v)\right)\,.$$

Therefore

$$-J^0\left(L(v); L(x-v)\right) \leq \liminf_{n\to\infty} \left(-J^0\left(L(v_n); L(x-v_n)\right)\right).$$

So, if $v_n \in G(x)$ and $v_n \rightharpoonup v$ then

$$\begin{aligned}&\langle Av - f, v - x\rangle - J^0\left(L(v); L(x-v)\right) + \Phi(v) - \Phi(x) \leq \\ &\liminf \left\{\langle Av_n - f, v_n - x\rangle - J^0\left(L(v_n); L(v-v_n)\right) + \Phi(v_n) - \Phi(x)\right\} \leq 0\,,\end{aligned}$$

which shows that $v \in G(x)$. Since K is bounded, it follows that $G(x)$ is weakly compact. This implies that

$$\bigcap_{x\in K\cap D(\Phi)} G(x) \neq \emptyset\,,$$

provided that the family $\{G(x); x \in K \cap D(\Phi)\}$ has the finite intersection property. We may conclude by using the KKM principle after showing that G is a KKM-mapping. Suppose by contradiction that there exist $x_1, \cdots, x_n \in K \cap D(\Phi)$ and $y_0 \in \operatorname{Conv}\{x_1, \cdots, x_n\}$ such that $y_0 \notin \bigcup_{i=1}^n G(x_i)$. Then

$$\langle Ay_0 - f, y_0 - x_i\rangle + \Phi(y_0) - \Phi(x_i) - J^0\left(L(y_0); L(x_i - y_0)\right) > 0\,, \quad \forall i = 1, \cdots, n\,.$$

Therefore

$$x_i \in \Lambda := \{x \in X; \langle Ay_0 - f, y_0 - x\rangle + \Phi(y_0) - \Phi(x) - J^0\left(L(y_0); L(x - y_0)\right) > 0\}\,,$$

for all $i \in \{1, \cdots, n\}$. The set Λ is convex and thus $y_0 \in \Lambda$, leading to an obvious contradiction. So,

$$\bigcap_{x\in K\cap D(\Phi)} G(x) \neq \emptyset\,.$$

This yields an element $u \in K \cap D(\Phi)$ such that, for any $v \in K \cap D(\Phi)$,

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0\left(L(u); L(v-u)\right) \geq 0\,.$$

This inequality is trivially satisfied if $v \notin D(\Phi)$ and the conclusion follows. $\square$

We may now derive a result applicable to the inequality problem (6). Indeed, suppose that the above hypotheses are satisfied and set $Y = L^p(T, \mathbf{R}^m)$. Let $J : Y \to \mathbf{R}$ be the function defined by

$$J(u) = \int_T j(x, u(x))d\mu\,. \tag{8}$$

The conditions (2) or (3)-(4) on j ensure that J is locally Lipschitz on Y and

$$\int_T j^0(x, u(x); v(x))d\mu \geq J^0(u; v)\,, \quad \forall u, v \in X\,.$$

It follows that

$$\int_T j^0(x, \gamma(u(x)); \gamma(v(x)))d\mu \geq J^0(\gamma(u); \gamma(v)), \quad \forall u, v \in X. \tag{9}$$

It results that if $u \in K$ is a solution of (7) then u solves the inequality problem (6), too. The following result follows.

Theorem 1 *Assume that the hypotheses of Lemma 1 are fulfilled for $Y = L^p(T, \mathbf{R}^m)$ and $L = \gamma$. Then the problem (6) has at least a solution.*

In order to establish a variant of Lemma 1 for monotone and hemicontinuous operators we need the following result which is due to Mosco (see [9]):

Mosco's Theorem. *Let K be a nonempty convex and compact subset of a topological vector space X. Let $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $D(\Phi) \cap K \neq \emptyset$. Let $f, g : X \times X \to \mathbf{R}$ be two functions such that*
(i) $g(x, y) \leq f(x, y)$, for every $x, y \in X$;
(ii) the mapping $f(\cdot, y)$ is concave, for any $y \in X$;
(iii) the mapping $g(x, \cdot)$ is lower semicontinuous, for every $x \in X$.

Let λ be an arbitrary real number. Then the following alternative holds: either
- there exists $y_0 \in D(\Phi) \cap K$ such that $g(x, y_0) + \Phi(y_0) - \Phi(x) \leq \lambda$, for any $x \in X$,
or
- there exists $x_0 \in X$ such that $f(x_0, x_0) > \lambda$.

We notice that two particular cases of interest for the above result are if $\lambda = 0$ or $f(x, x) \leq 0$, for every $x \in X$.

Lemma 2 *Let K be a nonempty, bounded, closed subset of the real reflexive Banach space X, and $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a convex and lower semicontinuous function such that (5) holds. Consider a linear subspace Y of $X^\star$ such that there exists a linear and compact mapping $L : X \to Y$. Let $J : Y \to \mathbf{R}$ be a locally Lipschitz function. Suppose in addition that the operator $A : X \to X^\star$ is monotone and hemicontinuous.*

Then for each $f \in X^\star$, the inequality problem (7) has at least a solution .

Proof. Set

$$g(x, y) = \langle Ax - f, y - x \rangle - J^0(L(y); L(x) - L(y))$$

and

$$f(x, y) = \langle Ay - f, y - x \rangle - J^0(L(y); L(x) - L(y)).$$

The monotonicity of A implies that

$$g(x,y) \le f(x,y)\,, \qquad \forall x,y \in X\,.$$

The mapping $x \longmapsto f(x,y)$ is concave while the mapping $y \longmapsto g(x,y)$ is weakly lower semicontinuous. Applying Mosco's Theorem with $\lambda = 0$, we obtain the existence of $u \in K \cap D(\Phi)$ satisfying

$$g(w,u) + \Phi(u) - \Phi(w) \le 0\,, \qquad \forall w \in K\,,$$

that is

$$\langle Aw - f, w-u\rangle + \Phi(w) - \Phi(u) + J^0(L(u); L(w-u)) \ge 0\,, \qquad \forall w \in K\,. \tag{10}$$

We use in what follows an argument which is in the same spirit as that used in the proof of Minty's Lemma (see [5, Lemma III.1.5]). Fix $v \in K$ and set $w = u + \lambda(v-u) \in K$, for $\lambda \in [0,1)$. So, by (10),

$$\lambda\,\langle A(u+\lambda(v-u)) - f, v-u\rangle + \Phi(\lambda v + (1-\lambda)u)) - \Phi(u) + J^0(L(u); \lambda L(v-u)) \ge 0\,.$$

Using the convexity of Φ, the fact that $J^0(u;\cdot)$ is positive homogeneous (see [1], p. 103) and dividing then by $\lambda > 0$ we find

$$\langle A(\lambda v + (1-\lambda)u) - f, v-u\rangle + \Phi(v) - \Phi(u) + J^0(L(u); L(v-u)) \ge 0\,.$$

Now, taking $\lambda \to 0$ and using the hemicontinuity of A we find that u solves (7). □

The analogue of Theorem 1 for monotone and hemicontinuous operators can now be stated as follows:

Theorem 2 *Assume that the hypotheses of Lemma 2 are fulfilled for $Y = L^p(T, \mathbf{R}^m)$ and $L = \gamma$. Then the inequality problem (6) admits at least a solution.*

3 Coercive variational-hemivariational inequalities

We observe that if j satisfies conditions (1) and (2) then, by the Cauchy-Schwarz Inequality,

$$\left|\int_T j^0(x, \gamma(u(x)); \gamma(v(x)))d\mu\right| \le \int_T k(x)|\gamma(v(x))|d\mu \le |k|_{p'} \cdot |\gamma(v)|_p \le C\,|k|_{p'}\,\|v\|\,, \tag{11}$$

where $|\cdot|_p$ denotes the norm in the space $L^p(T, \mathbf{R}^m)$ and $\|\cdot\|$ stands for the norm in X. On the other hand, if j satisfies conditions (1), (3) and (4) then

$$|j^0(x, \gamma(u(x)); \gamma(v(x)))| \le C\,(1 + |\gamma(u(x))|^{p-1})\,|\gamma(v(x))|$$

and thus

$$|\int_T j^0(x,\gamma(u(x));\gamma(v(x)))d\mu| \leq C\left(|\gamma(v)|_1 + |\gamma(u)|_{p-1}^p|\gamma(v)|_p\right) \leq C_1\,\|v\| + C_2\,\|u\|^{p-1}\|v\|\,, \quad (12)$$

for some suitable constants $C_1, C_2 > 0$. We discuss in this framework the solvability of coercive variational-hemivariational inequalities.

Theorem 3 *Let K be a nonempty closed convex subset of X, $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous function such that $K \cap D(\Phi) \neq \emptyset$ an $A : X \to X^\star$ and operator such that the mapping $v \longmapsto \langle Av, v - x\rangle$ is weakly lower semicontinuous, for all $x \in K$. The following hold*

(i) If j satisfies conditions (1) and (2), and if there exists $x_0 \in K \cap D(\Phi)$ such that

$$\frac{\langle Aw, w - x_0\rangle + \Phi(w)}{\|w\|} \to +\infty\,, \qquad \text{as } \|w\| \to +\infty \quad (13)$$

then for each $f \in X^\star$, there exists $u \in K$ such that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + \int_T j^0(x,\gamma(u(x));\gamma(v(x)) - \gamma(u(x)))d\mu \geq 0\,, \quad \forall v \in K\,. \quad (14)$$

(ii) If j satisfies conditions (1), (3) and (4) and if there exist $x_0 \in K \cap D(\Phi)$ and $\theta \geq p$ such that

$$\frac{\langle Aw, w - x_0\rangle}{\|w\|^\theta} \to +\infty\,, \qquad \text{as } \|w\| \to +\infty \quad (15)$$

then for each $f \in X^\star$, there exists $u \in K$ satisfying (14).

Proof. There exists a positive integer n_0 such that

$$x_0 \in K_n := \{x \in K; \|x\| \leq n\}\,, \qquad \forall n \geq n_0\,.$$

Applying Lemma 1 with J as defined in (8) we find some $u_n \in K_n$ such that, for every $n \geq n_0$ and any $v \in K_n$,

$$\langle Au_n - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) + J^0(\gamma(u_n);\gamma(v) - \gamma(u_n)) \geq 0\,. \quad (16)$$

We claim that the sequence (u_n) is bounded. Suppose by contradiction that $\|u_n\| \to +\infty$. Then, passing eventually to a subsequence, we may assume that

$$v_n := \frac{u_n}{\|u_n\|} \rightharpoonup v\,.$$

Setting $v = x_0$ in (16) and using (9), we obtain

$$\begin{aligned} \langle Au_n, u_n - x_0\rangle + \Phi(u_n) \leq \ & \Phi(x_0) + \langle f, u_n - x_0\rangle + J^0(\gamma(u_n); \gamma(x_0) - u_n)) \leq \\ & \Phi(x_0) + \langle f, u_n - x_0\rangle + \left| \int_T j^0(x, \gamma(u_n); \gamma(x_0 - u_n)) d\mu \right| . \end{aligned} \tag{17}$$

Case (i). Using (11) we obtain

$$\langle Au_n, u_n - x_0\rangle + \Phi(u_n) \leq \Phi(x_0) + \langle f, u_n - x_0\rangle + c|k|_{p'} \, \|u_n - x_0\|$$

and thus

$$\frac{\langle Au_n, u_n - x_0\rangle + \Phi(u_n)}{\|u_n\|} \leq \frac{\Phi(x_0)}{\|u_n\|} + \langle f, v_n - x_0\|u_n\|^{-1}\rangle + c|k|_{p'} \, \|v_n - x_0\|u_n\|^{-1}\| . \tag{18}$$

Passing to the limit as $n \to \infty$ we observe that the left-hand term in (18) tends to $+\infty$ while the right-hand term remains bounded which yields a contradiction.

Case (ii). The function Φ being convex and lower semicontinuous, we may apply the Hahn-Banach separation theorem to find that

$$\Phi(x) \geq \langle \alpha, x\rangle + \beta , \qquad \forall x \in X ,$$

for some $\alpha \in X^\star$ and $\beta \in \mathbf{R}$. This means that

$$\Phi(x) \geq -\|\alpha\|_\star \, \|x\| + \beta , \qquad \forall x \in X .$$

From (17) and (12) we deduce that

$$\langle Au_n, u_n - x_0\rangle \leq \Phi(x_0) + \|\alpha\|_\star \, \|u_n\| - \beta + \langle f, u_n - x_0\rangle + C_1 \, \|u_n - x_0\| + C_2 \, \|u_n\|^{p-1}\|u_n - x_0\| .$$

Thus

$$\begin{aligned} \frac{\langle Au_n, u_n - x_0\rangle}{\|u_n\|^\theta} \leq \ & \|\alpha\|_\star \, \|u_n\|^{1-\theta} + (\Phi(x_0) - \beta)\|u_n\|^{-\theta} + \langle f, v_n\|u_n\|^{1-\theta} - x_0\|u_n\|^{-\theta}\rangle + \\ & C_1\|v_n\|u_n\|^{1-\theta} - x_0\|u_n\|^{-\theta}\| + C_2\|v_n - x_0\|u_n\|^{-1}\| \cdot \|u_n\|^{p-\theta} \end{aligned}$$

and taking the limit as $n \to \infty$ we obtain a contradiction, since $\theta \geq p \geq 1$.

Thus in both cases (i) and (ii), the sequence $\{u_n\}$ is bounded. This implies that, up to a subsequence, $u_n \rightharpoonup u \in K$. Let $v \in K$ be given. For all n large enough we have $v \in K_n$ and thus by (16),

$$\langle Au_n - f, u_n - v\rangle + \Phi(u_n) - \Phi(v) - J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \leq 0. \tag{19}$$

Passing to the limit as $n \to \infty$ we obtain

$$\langle Au - f, u - v\rangle \le \liminf_{n\to\infty} \langle Au_n - f, u_n - v\rangle$$

$$\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$$

$$\gamma(u) = \lim_{n\to\infty} \gamma(u_n)$$

and

$$-J^0(\gamma(u); \gamma(v) - \gamma(u)) \le \liminf_{n\to\infty} \left(-J^0(\gamma(u_n); \gamma(v) - \gamma(u_n))\right).$$

Taking the inferior limit in (19) we obtain

$$\langle Au - f, u - v\rangle + \Phi(u) - \Phi(v) - J^0(\gamma(u); \gamma(v) - \gamma(u)) \le 0.$$

Since v has been chosen arbitrarily we obtain

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \ge 0, \qquad \forall v \in K.$$

Using now again (9) we conclude that u solves (14). □

The following result gives a corresponding variant for monotone hemicontinuous operators.

Theorem 4 *Let K be a nonempty closed convex subset of X, $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ a proper convex and lower semicontinuous function such that $D(\Phi) \cap K \neq \emptyset$. Let $A : X \to X^\star$ be a monotone and hemicontinuous operator. Assume (13) or (15) as in Theorem 3. Then the conclusions of Theorem 3 hold true.*

Proof. Using Lemma 2 we find a sequence $u_n \in K_n$ such that

$$\langle Au_n - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) + J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \ge 0, \qquad \forall v \in K_n. \tag{20}$$

As in the proof of Theorem 3 we justify that $\{u_n\}$ is bounded and thus, up to a subsequence, we may assume that $u_n \rightharpoonup u$. By (20) and the monotonicity of A we deduce that

$$\langle Av - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) + J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \ge 0.$$

Let $v \in K$ be given. For n large enough we obtain

$$\langle Av - f, u_n - v\rangle + \Phi(u_n) - \Phi(v) - J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \le 0$$

and taking the inferior limit we obtain

$$\langle Av - f, u - v\rangle + \Phi(u) - \Phi(v) - J^0(\gamma(u); \gamma(v) - \gamma(u)) \leq 0.$$

Since v has been chosen arbitrarily it follows that

$$\langle Av - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \geq 0, \qquad \forall v \in K.$$

Using now the same argument as in the proof of Lemma 2 we obtain that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \geq 0, \qquad \forall v \in K$$

and the conclusion follows now by (9). □

4 Noncoercive variational-hemivariational inequalities

In order to treat noncoercive cases we use in this Section a minimax approach for studying the inequality problem (7) (in particular, (6)). To this end we present the necessary background of nonsmooth critical point theory developed in Motreanu-Panagiotopoulos ([10], Chapter III).

Definition 1 (Definition 3.1 in Motreanu-Panagiotopoulos [10]). *Let X be a real Banach space, let $F : X \to \mathbf{R}$ be a locally Lipschitz function and let $G : X \to \mathbf{R} \cup \{+\infty\}$ be a proper (i.e., $\not\equiv +\infty$), convex and lower semicontinuous function. An element $u \in X$ is called a critical point of the functional $I = F + G : X \to \mathbf{R} \cup \{+\infty\}$ if the inequality below holds*

$$F^0(u; v - u) + G(v) - G(u) \geq 0, \ \forall \ v \in X.$$

Definition 2 (Definition 3.2 in Motreanu-Panagiotopoulos [10]). *The functional $I = F + G : X \to \mathbf{R} \cup \{+\infty\}$ as in Definition 1 is said to satisfy the Palais - Smale condition if every sequence $\{u_n\} \subset X$ for which $I(u_n)$ is bounded and*

$$F^0(u_n; v - u_n) + G(v) - G(u_n) \geq -\varepsilon_n \|v - u_n\|, \ \ \forall v \in X,$$

for a sequence $\{\varepsilon_n\} \subset \mathbf{R}^+$ with $\varepsilon_n \to 0$, contains a strongly convergent subsequence in X.

Remark. Definitions 1 and 2 extend and unify the nonsmooth critical point theories due to Chang [1] and Szulkin [19]. Precisely, if $G = 0$ Definitions 1 and 2 reduce to the corresponding definitions of Chang [1], while if $F \in C^1(X, \mathbf{R})$ Definitions 1 and 2 coincide with those in Szulkin [19].

Mountain Pass Theorem. (Corollary 3.2 in Motreanu-Panagiotopoulos [10]) *Let* $I = F + G : X \to \mathbf{R} \cup \{+\infty\}$ *be a functional as in Definition 1 which satisfies the Palais-Smale condition in the sense of Definition 2. Assume that there exist a number* $\rho > 0$ *and a point* $e \in X$ *with* $\|e\|_X > \rho$ *such that*

$$\inf_{\|u\|_X = \rho} I > \max\{I(0), I(e)\}.$$

Then the number

$$c = \inf\{\sup_{t\in[0,1]} I(f(t)) : \quad f \in C([0,1], X),\ f(0) = 0,\ f(1) = e\} \geq \inf_{\|u\|_X=\rho} I$$

is a critical value of I*, i.e., there exists* $u \in X$ *such that* $I(u) = c$ *and* u *is a critical point of* I *in the sense of Definition 1.*

Let us describe now the abstract functional framework of our variational approach in studying the inequality problem (7) without the assumptions of boundedness for set K or of coerciveness as in Theorem 3. Let X and Y be Banach spaces, with X reflexive, and let $L : X \to Y$ be a linear compact operator. Consider the functionals $E \in C^1(X, \mathbf{R})$ (in (7) we will take $A := E' : X \to X^*$), $\Phi : X \to \mathbf{R}$ convex, lower semicontinuous, Gâteaux differentiable and $J : Y \to \mathbf{R}$ locally Lipschitz. Given a closed convex cone K of X, with $0 \in K$, let I_K denote the indicator function of K. We apply the aforementioned nonsmooth version of Mountain Pass Theorem for the following choices: $F := E + J \circ L$, $G := \Phi + I_K$ and thus $I = F + G$.

The following result follows readily from Definition 1.

Lemma 3 *Every critical point* $u \in X$ *of the functional* I *in the sense of Definition 1 is a solution to problem (7) with* $A = E'$.

Lemma 4 *Assume in addition that the following hypotheses are satisfied:*
(H1) There exist positive constants a_0, a_1, α *with* $\alpha < a_0$ *such that*

$$E(v) + \Phi(v) + J(Lv) - \alpha(\langle E'(v) + \Phi'(v), v\rangle + J^0(Lv; Lv))$$

$$\geq a_0\|v\| - a_1,\ \forall\ u \in K,$$

and
(H2) If $\{u_n\}$ *is a sequence in* K *provided* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty}\langle E'(u_n), u_n - u\rangle \leq 0$ *for some* $u \in X$*, then* $\{u_n\}$ *contains a subsequence denoted again by* $\{u_n\}$ *with* $u_n \to u$ *in* X.
Then the functional I *satisfies the Palais-Smale condition in the sense of Definition 2.*

Proof. Let $\{u_n\}$ be a sequence in X with the properties required in Definition 1. In particular, we know that $\{u_n\} \subset K$ and there exist a constant $M > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbf{R}^+$ with $\varepsilon_n \to 0$ such that

$$|I(u_n)| \leq M, \quad \forall n \geq 1,$$

and

$$\langle E'(u_n), v - u_n \rangle + J^0(Lu_n; Lv - Lu_n) + \Phi(v) - \Phi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in K.$$

Using the convexity and the Gâteaux differentiabiliy of Φ, setting $v = (1+t)u_n$, with $t > 0$, in the inequality above and then letting $t \to 0$ one obtains that

$$\langle E'(u_n) + \Phi'(u_n), u_n \rangle + J^0(Lu_n; Lu_n) \geq -\varepsilon_n \|u_n\|, \quad \forall n \geq 1.$$

The inequalities above ensure that for n sufficiently large (so that $\varepsilon_n \leq 1$) one has

$$M + \alpha \|u_n\|$$

$$\geq E(u_n) + \Phi(u_n) + J(Lu_n) - \alpha[\langle E'(u_n) + \Phi'(u_n), u_n \rangle + J^0(Lu_n; Lu_n)].$$

Here α denotes the positive constant entering assumption $(H1)$. Then on the basis of condition $(H1)$ we deduce that the sequence $\{u_n\}$ is bounded in X.

Consequently, the sequence $\{u_n\}$ contains a subsequence again denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ in X and $Lu_n \to Lu$ in Y for some $u \in K$. On the other hand if we set $v = u$, we derive that

$$\langle E'(u_n), u - u_n \rangle + J^0(Lu_n; Lu - Lu_n) + \Phi(u) - \Phi(u_n) \geq -\varepsilon_n \|u - u_n\|.$$

Since J^0 is upper semicontinuous and Φ is lower semicontinuous, this yields that

$$\limsup_{n \to \infty} \langle E'(u_n), u_n - u \rangle \leq 0.$$

Assumption $(H2)$ completes the proof. $\square$

The main result of this Section is stated below.

Theorem 5 *Assume* $(H1)$, $(H2)$,
$(H3)$ *There exist an element* $\overline{u} \in K \setminus \{0\}$ *satisfying* $\|\overline{u}\| > a_1/a_0$, *for the constants* a_0, a_1 *in* $(H1)$, *and* $E(\overline{u}) + \Phi(\overline{u}) + J(\overline{u}) \leq 0$,
and
$(H4)$ *There exist a constant* $\rho > 0$ *such that*

$$\inf_{\|v\|=\rho} (E(v) + \Phi(v) + J(v)) > E(0) + \Phi(0) + J(0).$$

Then problem (7) with $A = E'$ *admits at least a solution* $u \in K \setminus \{0\}$.

Proof. Let us apply the nonsmooth version of Mountain Pass Theorem to our functional I. Lemma 4 establishes that I satisfies the Palais-Smale condition in the sense of Definition 2.

The calculus with generalized gradients (see Clarke [2]) shows that

$$\partial_t(t^{-\frac{1}{\alpha}}(E+\Phi)(tu)+t^{-\frac{1}{\alpha}}J(tLu))$$

$$\subset -\frac{1}{\alpha}t^{-\frac{1}{\alpha}-1}(E+\Phi)(tu)+t^{-\frac{1}{\alpha}}\langle(E'+\Phi')(tu),u\rangle$$

$$-\frac{1}{\alpha}t^{-\frac{1}{\alpha}-1}J(tLu)+t^{-\frac{1}{\alpha}}\partial J(tLu)u,\ \forall t>0,\ \forall u\in X,$$

where the notation ∂_t stands for the generalized gradient with respect to t. Lebourg's mean value theorem allows to find some $\tau=\tau(u)\in(1,t)$ such that

$$t^{-\frac{1}{\alpha}}(E(tu)+\Phi(tu)+J(tLu))-(E(u)+\Phi(u)+J(u))$$

$$\in\frac{1}{\alpha}\tau^{-\frac{1}{\alpha}-1}[\alpha(\langle E'(\tau u)+\Phi'(\tau u),\tau u\rangle+\partial J(\tau u)\tau u)$$

$$-(E(\tau u)+\Phi(\tau u)+J(\tau u))](t-1),\quad \forall t>1,\ \forall u\in X.$$

Combining with assumption $(H1)$ it follows that

$$t^{-\frac{1}{\alpha}}(E(tu)+\Phi(tu)+J(tLu))-(E(u)+\Phi(u)+J(u))$$

$$\le\frac{1}{\alpha}\tau^{-\frac{1}{\alpha}-1}(-a_0\tau\|u\|+a_1)(t-1),\quad \forall t>1,\ \forall u\in K.$$

It is then clear from assumption $(H3)$ that one can write

$$I(t\overline{u})=E(t\overline{u})+\Phi(t\overline{u})+J(t\overline{u})\le t^{\frac{1}{\alpha}}[E(\overline{u})+\Phi(\overline{u}))+J(\overline{u})],\quad \forall t>1.$$

This fact in conjunction with assumption $(H3)$ leads to the conclusion that

$$\lim_{t\to+\infty}I(t\overline{u})=-\infty.$$

Then assumption $(H4)$ enables us to apply the nonsmooth version of Mountain Pass Theorem for $e=t\overline{u}$, with a sufficiently large positive number t. According to Mountain Pass Theorem the functional I possesses a nontrivial critical point $u\in X$ in the sense of Definition 1. Finally, Lemma 3 shows that u is a (nontrivial) solution of problem (7) with $A=E'$. The proof of Theorem 5 is thus complete. $\square$

We end this Section with an example of application of Theorem 5 in the case of variational-hemivariational inequality (6). For the sake of simplicity we consider a uniformly convex Banach space X, a convex closed cone K in X with $0 \in K$, $f = 0$, $\Phi = 0$ and a self-adjoint linear continuous operator $A : X \to X^*$ satisfying $\langle Av, v\rangle \geq c_0\|v\|^2$, for all $v \in X$, with a constant $c_0 > 0$.

Assume that the function $j : T \times \mathbf{R}^m \to \mathbf{R}$ verifies the conditions (1), (3), (4) with $p > 2$, as well as the following assumptions of Ambrosetti–Rabinowitz type:

(i) there exist constants $0 < \alpha < 1/2$ and $c \in \mathbf{R}$ such that

$$j(x, y) \geq \alpha j_y^0(x, y; y) + c, \quad \text{for a.e. } x \in T, \ \forall y \in \mathbf{R}^m;$$

(ii) $\liminf\limits_{y\to 0} \frac{1}{|y|^2} j(x, y) \geq 0$ uniformly with respect to $x \in T$, and $j(x, 0) = 0$ a.e. $x \in T$;

(iii) there exists an element $u_0 \in K \setminus \{0\}$ such that

$$\liminf_{t\to\infty} \left[\frac{1}{2}\langle Au_0, u_0\rangle t^2 + \int_T j(x, tu_0(x))dx\right] < 0.$$

Let us apply Theorem 5 for the functional J given by (8) and $E(v) = (1/2)\langle Av, v\rangle$, $\forall v \in X$. We see that hypotheses (i) and (ii) imply $(H1)$ and $(H4)$, respectively. Taking $\overline{u} = tu_0$ for $t > 0$ sufficiently large, we get $(H3)$ from (iii). It is straightforward to check that condition $(H2)$ holds true. Therefore Theorem 5 yields a nontrivial solution of variational-hemivariational inequality (6) in our setting.

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